Algebraic Topology 2004 **Example Sheet 1**

1. Let $a\colon S^n \to S^n$ be the antipodal map ($a(x) = -x$). Prove that a is homotopic to the identity if n is odd. (Hint: Consider the case $n = 1$ first.) [Later we'll see that n is not homotopic to the identity if n is even.]

2. Let $f\colon S^1 \to S^1$ be a map that is not homotopic to the identity map.

   (a) Show that $f(x) = -x$ for some $x \in S^1$.

   (b) Show that $f(y) = y$ for some $y \in S^1$.

3. Let X be a contractible space and let Y be any space. Show that:

   (a) X is path connected.

   (b) $X \times Y$ is homotopy equivalent to Y.

   (c) Any two maps from Y to X are homotopic.

   (d) If Y is path connected, any two maps from X to Y are homotopic.

4. Let X be the subset of $\mathbb{R}^2$ consisting of the point $(0, 1)$ together with all the points on the line segment connecting $(0, 1)$ to $(0, 0)$ and the line segments connecting $(0, 1)$ with each of the points $(1/n, 0)$ for $n = 1, 2, 3, \ldots$.

   (a) Show that X is contractible.

   (b) Show that $A = \{(0, 1)\} \subset X$ is a deformation retract.

   (c) Show that $B = \{(0, 0)\} \subset X$ is not a deformation retract – any homotopy between the identity and the constant map sending all of X to $(0, 0)$ must "move" the point $(0, 0)$.

5. Show that the torus minus one point, the Klein bottle minus one point, and the plane ($\mathbb{R}^2$) minus two points are all homotopy equivalent to $S^1 \vee S^1$, the one-point union of two copies of S^1. (Hint: Draw pictures embedding $S^1 \vee S^1$ in each space and describe the homotopies in words rather than in formulas.)

6. Show that the following are equivalent for a path connected space X:

   (i) X is simply connected.

   (ii) Every continuous map $f\colon S^1 \to X$ can be extended over B^2.

   (iii) For any two points $a, b \in X$, any two paths from a to b are homotopic.

7. Let (X, x) and (Y, y) be based spaces. Show that $\pi_1(X \times Y, (x, y))$ is isomorphic to $\pi_1(X, x) \times \pi_1(Y, y)$.

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8. Let G be a space which has a continuous multiplication $m: G \times G \to G$ and a point $e \in G$ that acts as the identity, $m(e, g) = g = m(g, e)$ for all $g \in G$. (For example, G might be a topological group but associativity and inverses will play no role in the question). Given loops α and β based at e, show that the loop γ defined by $\gamma(s) = m(\alpha(s), \beta(s))$ is homotopic to both $\beta\alpha$ and $\alpha\beta$. Conclude that $\pi_1(G, e)$ is abelian.

9. Show that the cylinder and the Möbius band both have fundamental group isomorphic to $\mathbb{Z}$.

10. Prove that no two of the spaces S^2, S^1, or $S^0 = \{-1, 1\} \subset \mathbb{R}$ are homotopy equivalent. Prove that no two of the spaces $\mathbb{R}^3$, $\mathbb{R}^2$, or $\mathbb{R}$ are homeomorphic.

11. Prove the Intermediate Value Theorem from Real Analysis (of one variable), that a continuous function from a closed interval to the real numbers takes on every value between the values of its endpoints. Relate this to the Brouwer Fixed Point Theorem.

12. Consider the polynomials with complex coefficients

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \qquad \text{and} \qquad q(z) = z^n.$$

Let $C_r = \{z : |z| = r\}$. Show that for sufficiently large values of r,

$$p|_{C_r}: C_r \to \mathbb{C} - \{0\} \qquad \text{and} \qquad q|_{C_r}: C_r \to \mathbb{C} - \{0\}$$

are homotopic. Deduce the Fundamental Theorem of Algebra, that $p(z) = 0$ for some $z \in \mathbb{C}$.

13. Draw or describe an n-fold cover of the two-holed torus (the double innertube) for n each of your three favorite pairwise relatively prime positive integers (e.g., $n = 2, 3, 5$).

14. Construct a covering map from $\mathbb{R}^2$ to the Klein bottle and use it to identify the fundamental group with the nonabelian group whose elements are pairs of integers (m, n) and whose multiplication is

$$(m, n) * (p, q) = (m + (-1)^n p, n + q).$$

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15. Describe the universal cover of $S^1 \vee S^2$, the one-point union of S^1 and S^2, and show that the fundamental group of $S^1 \vee S^2$ is isomorphic to $\mathbb{Z}$.

16. Let G be a path connected and locally path connected topological group, and let $f\colon \Gamma \to G$ be a covering map. Choose an element ϵ in Γ in the preimage of the identity element e of G.

    (a) Show that Γ there exists a unique topological group structure on Γ with ϵ the identity and f a homomorphism.

    (b) Show that the kernel of f is central (every element of Γ commutes with each element of the kernel).

17. Let (X, x) be a based topological space and let $f\colon X \to X$ be a based map that is homotopic to the identity. Show that the induced map on π_1 is conjugation by the homotopy class of the loop that is the path taken by the basepoint in a homotopy from f to the identity.

18. Let $C_1(-2, 0)$, $C_1(0, 0)$, and $C_1(2, 0)$ denote the circles of radius 1 centered on $(-2, 0)$, $(0, 0)$, and $(2, 0)$. Let X be the union of all three circles and let Y be the union of the last two (so Y is homeomorphic to $S^1 \vee S^1$).

    (a) Construct a map $X \to Y$ that is a (twofold) covering map. (Hint: think about an action on X, e.g., the one from multiplication by -1 on $\mathbb{C}$.)

    (b) Use $(1, 0)$ as a basepoint for Y, let α be the counterclockwise loop around $C_1(2, 0)$ and let β be the counterclockwise loop around $C_1(0, 0)$. Use path lifting to X to prove that $\beta\alpha \neq \alpha\beta$.

    (c) Describe a based self-map $f\colon Y \to Y$ that is homotopic to the identity but does not induce the identity map on π_1.

End of Example Sheet 1.

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