Algebraic Geometry Example Sheet 1: Lent 2025

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `hk439@cam.ac.uk`. In all questions, k is an algebraically closed field of characteristic 0.

1. Describe the closed sets in $\mathbb{A}^2$ with the Zariski topology. Describe the closed sets in $\mathbb{A}^1 \times \mathbb{A}^1$ with the product topology, where each factor is given the Zariski topology.

2. Show that any non-empty open subset of an irreducible algebraic set is dense and irreducible.

3. Recall that a collection of open sets $\mathcal{B} \subset \tau$ in a topology τ is a *basis* for τ if every open set in τ is a union of elements of $\mathcal{B}$. Show that if X is an affine variety, the collection of distinguished open sets $\{D(f) := X \setminus Z(f) \mid f \in A(X)\}$ forms a basis for the Zariski topology on X.

4. Let $X \subset \mathbb{A}^n$ be an affine variety. Suppose $X = X_1 \cup \cdots \cup X_r$ and $X = X_1' \cup \cdots \cup X_s'$ are two decompositions of X into irreducible components. Assume that $X_i \not\subset X_j$ and $X_i' \not\subset X_j'$ for any $i \neq j$. Prove that $r = s$ and that the two decompositions agree up to reordering; that is, irreducible decompositions are essentially unique.

5. Identify $\mathbb{A}^{mn}_{\mathbb{C}}$ with the set of complex $m \times n$ matrices. Prove that the subset $GL_n(\mathbb{C})$ of invertible matrices is Zariski dense in $\mathbb{A}^{n^2}$. An $m \times n$ matrix is said to have *full rank* if it has a $k \times k$ minor of non-vanishing determinant, where $k = \min\{m, n\}$. Show that the set of matrices of full rank is Zariski dense in $\mathbb{A}^{mn}$.

6. Let $f, g \in k[x, y]$ be polynomials which have no common factor. Show that there exist $u, v \in k[x, y]$ such that $uf + vg$ is a non-zero polynomial in $k[x]$. Now assume that f is irreducible, and show that any proper subvariety of $Z(f)$ is finite - that is, affine plane curves which do not share a component intersect in finitely many points. Deduce that $\mathbb{A}^2$ with the Zariski topology has the property that every open cover has a finite subcover.

7. Let $X \subset \mathbb{A}^2$ be the affine plane curve $X := Z(xy - 1)$. Show that X is not isomorphic to $\mathbb{A}^1$, and describe all morphisms $\mathbb{A}^1 \to X$.

8. Let $Y = Z(x^2 - yz, xz - x) \subset \mathbb{A}^3$. Show that Y has three irreducible components. Describe the three components and their corresponding prime ideals.

9. Show that if $X \subset \mathbb{A}^m, Y \subset \mathbb{A}^n$ are affine varieties, then $X \times Y \subset \mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{n+m}$ is also an affine variety. (More difficult: the product of irreducible varieties is irreducible).

10. Show that there are no non-constant morphisms from $\mathbb{A}^1$ to $E := Z(y^2 - x^3 + x) \subset \mathbb{A}^2$. [Hint: consider the images of x and y under the induced map $A(E) \to A(\mathbb{A}^1) = k[t]$, and use the fact that $k[t]$ is a UFD.]

The following exercises are more difficult.

11. Let $X \subset \mathbb{A}^3$ be the set $\{(t^3, t^4, t^5) \mid t \in k\}$. Show that X is an affine variety, and determine $I(X)$ [Hint: $I(X)$ can be generated by three elements.] Show that $I(X)$ cannot be generated by two elements.

12. Let $\pi : \mathbb{A}^3 \to \mathbb{A}^1$ be the projection onto the third coordinate. For each point $z \in \mathbb{A}^1$, the set-theoretic preimage $\pi^{-1}(z)$ is isomorphic to $\mathbb{A}^2$. Construct a variety $X \subset \mathbb{A}^3$ with the property that if $z \neq 0$ then $\pi^{-1}(z) \cap X$ is a union of two intersecting lines in $\pi^{-1}(z)$, but $\pi^{-1}(0) \cap X$ is a union of two parallel lines in $\pi^{-1}(0)$.

13. Let $V \subset \mathbb{A}^3$ be the union of the x, y, and z axes. Let $W \subset \mathbb{A}^2$ be the union of the x-axis, y-axis, and and the line $x = y$. Calculate generators for $I(V)$ and $I(W)$, and show that V is not isomorphic to W. [Hint: if p is a point on V, consider the ideal $\mathfrak{m}_p$ of elements of the coordinate ring $A(V)$ which vanish at p. The quotient $\mathfrak{m}_p/\mathfrak{m}_p^2$ is a k-vector space. What are the possible dimensions of this vector space for $p \in V$? What about for $p \in W$?]