Part II Algebraic Geometry

Example Sheet III, 2024

(For all questions, assume k is algebraically closed. Further, you can assume the characteristic is not equal to 2 if necessary.)

1. Determine the singular points of the surface in $\mathbb{P}^3$ defined by the polynomial $x_1x_2^2 - x_3^3 \in k[x_0, \ldots, x_3]$. Find the dimension of the tangent space at all the singularities.
2. Let $f, g : X \to Y$ be morphisms between algebraic varieties, and suppose there is a non-empty open subset $U \subseteq X$ such that $f|_U = g|_U$. Show $f = g$. [Hint: First reduce to the case $Y = \mathbb{P}^n$, and show that the map $f \times g : X \to \mathbb{P}^n \times \mathbb{P}^n$ is a morphism, where $f \times g(x) = (f(x), g(x))$. Next consider the diagonal $\Delta = \{(y, y) \mid y \in \mathbb{P}^n\} \subseteq \mathbb{P}^n \times \mathbb{P}^n$.]
3. Let X and Y be algebraic varieties. Recall that in defining rational map, we considered pairs (U, f) where $U \subseteq X$ is an open subset and $f : U \to Y$ is a morphism. We defined a relation $(U, f) \sim (V, g)$ if $f|_{U\cap V} = g|_{U\cap V}$. Show this relation is an equivalence relation.
4. Let M_1 be the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}.$$

Show that the set of points

$$C := \{(a_0 : \cdots : a_n) \in \mathbb{P}^n \mid \mathrm{rank} M_2(a_0, \ldots, a_n) = 1\}$$

is isomorphic to $\mathbb{P}^1$. This embedding of $\mathbb{P}^1$ in $\mathbb{P}^n$ is called the *rational normal curve*. You have already seen the special case $n = 3$, where the rational normal curve is called the twisted cubic.

Let M_2 be the matrix

$$\begin{pmatrix} x_0 & x_2 & x_3 & \cdots & x_{n-1} \\ x_1 & x_3 & x_4 & \cdots & x_n \end{pmatrix}.$$

Show that the set of points

$$X := \{(a_0 : \cdots : a_n) \in \mathbb{P}^n \mid \mathrm{rank}\, M_1(a_0, \ldots, a_n) = 1\}$$

has a map $f : X \to \mathbb{P}^1$ (you do not need to show this map is a morphism, but be sure you *think* it is a morphism), and that for $p \in \mathbb{P}^1$ we have $f^{-1}(p) \cong \mathbb{P}^1$. The variety X is called the *rational normal scroll*. (A variety X with a morphism $X \to Y$ all of whose fibres are projective lines is called a *scroll*; this is a particular example of a scroll.)
5. Let $V \subset \mathbb{P}^2$ be defined by $x_1^2x_2 = x_0^3$.
   (a) Show that the formula $(u : v) \mapsto (u^2v : u^3 : v^3)$ defines a morphism $\phi : \mathbb{P}^1 \to V$.

(b) Write down a rational map $\psi : V \dashrightarrow \mathbb{P}^1$, a morphism on $U = V \setminus \{(0:0:1)\}$ which is inverse to ϕ on U. What is the geometric interpretation of ψ?

(c) Show that ψ does not extend to a morphism at $(0:0:1)$.

6. Let $V \subset \mathbb{P}^2$ be defined by $x_1^2 x_2 = x_0^2(x_0 + x_2)$. Find a surjective morphism $\phi : \mathbb{P}^1 \to V$ such that, for $P \in V$,

$$\#\phi^{-1}(P) = \begin{cases} 2 & \text{if } P = (0:0:1) \\ 1 & \text{otherwise} \end{cases}$$

Is there a rational map $\psi \colon V \dashrightarrow \mathbb{P}^1$, a morphism on $U = V \setminus \{(0:0:1)\}$, which coincides with ϕ^{-1} on U?

7. Let V be the quadric $Z(x_0x_3 - x_1x_2) \subset \mathbb{P}^3$, and H the plane $x_0 = 0$. Let $P = (1:0:0:0)$. Show that $\phi = (0 : x_1 : x_2 : x_3)$ defines a rational map $\phi \colon V \dashrightarrow H$ such that for $Q \in V$, the line PQ meets H in $\phi(Q)$ whenever this is defined.

Let $V_1 = V \cap \{x_1 = x_2\}$ and $L = H \cap \{x_1 = x_2\}$. Verify explicitly that ϕ induces an isomorphism $V_1 \xrightarrow{\cong} L$.

8. Consider the birational map $\phi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by $(x_1x_2 : x_0x_2 : x_0x_1)$, and let $P_0 = (1:0:0)$, $P_1 = (0:1:0)$ and $P_2 = (0:0:1)$ be the points, at which ϕ is not a morphism. Let $L \subset \mathbb{P}^2$ be a line. Show that ϕ gives a morphism $L \to \mathbb{P}^2$ such that:

(i) if $L \cap \{P_i\} = \emptyset$ then ϕ is an isomorphism of L with a conic in $\mathbb{P}^2$ which passes through all of the $\{P_i\}$;

(ii) if L contains just one P_i then ϕ is an isomorphism of L with another line in $\mathbb{P}^2$

Determine the effect of ϕ on the cubic C with defining polynomial $x_0(x_1^2 + x_2^2) + x_1^2x_2 + x_1x_2^2$. (Assume $\mathrm{char}(k) \neq 2$.) What happens to the singularity of C? Draw appropriate pictures.

9. (i) Let $\phi : X \to Y$ be a morphism of affine varieties. Using the definition of tangent space in terms of the derivatives of elements of the ideal, show that for all $p \in X$, there is a linear map

$$d\phi : T_pX \to T_{\phi(p)}Y.$$

(ii) In the situation of (i), if ϕ is defined by an m-tuple of polynomials $(\Phi_1, \ldots, \Phi_m) \in A(X)^m$, write $d\phi$ in terms of the Φ_i.

(iii) Now assume that X and Y are arbitrary varieties. Using the definition of Zariski tangent space, show (i) in this more general context. Show the your answer coincides with your answer in (i).

10. Let $Y \subseteq \mathbb{A}^3$ be the surface given by the equation $x_1^2 + x_2^2 + x_3^2 = 0$. Consider the blow-up $X \subseteq \mathbb{A}^3 \times \mathbb{P}^2$ of $\mathbb{A}^3$, with $\varphi : X \to \mathbb{A}^3$ the projection and $E = \varphi^{-1}(0)$. Recall that the *blowup* of Y is the closure of $\varphi^{-1}(Y) \setminus E$ in X. Describe the proper transform $\tilde{Y}$ of Y. Describe the fibres of the map $\varphi|_{\tilde{Y}} : \tilde{Y} \to Y$. Show that $\tilde{Y}$ is non-singular.