Part II Algebraic Geometry

Example Sheet II, 2024
(For all questions, assume k is algebraically closed.)

1. Show that the set of algebraic subsets of $\mathbf{P}^n$ forms a topology on $\mathbf{P}^n$.

2. Prove the "homogeneous Nullstellensatz," which says that if $I \subseteq S = k[x_0, \dots, x_n]$ is a homogeneous ideal and $f \in S$ is a homogeneous polynomial of degree greater than 0, and $f(p) = 0$ for all $p \in Z(I)$, then $f^q \in I$ for some $q > 0$. [Hint: Interpret this in the affine $n+1$-space whose coordinate ring is S.]

3. For a subset $X \subseteq \mathbf{P}^n$, define the ideal of X, $I(X)$, to be the ideal generated by homogeneous polynomials $f \in S$ such that $f(p) = 0$ for all $p \in X$. Let $I \subseteq S$ be a homogeneous ideal. Show that if $X = Z(I)$ is non-empty, then $I(X) = \sqrt{I}$. [Hint: You will need to show that $\sqrt{I}$ is generated by its homogeneous elements.]

   Show this may not be true if X is empty.

4. Show that if $I \subseteq k[x_0, \dots, x_n] = S$ is a homogeneous prime ideal and $Z(I) \neq \emptyset$, then $Z(I)$ is irreducible. Show that if $X \subseteq \mathbf{P}^n$ is a projective variety, then $I(X)$ is prime.

5. Given distinct points $P_0, \cdots, P_{n+1}$ in $\mathbf{P}^n$, no $(n+1)$ of which are contained in a hyperplane, show that homogeneous coordinates may be chosen on $\mathbf{P}^n$ so that $P_0 = (1\colon 0\colon \dots \colon 0)$, $\cdots$, $P_n = (0\colon \dots \colon 0\colon 1)$ and $P_{n+1} = (1\colon 1\colon \dots \colon 1)$. [This generalises to arbitrary n a result you are very familiar with when $n = 1$.]

6. Given hyperplanes $H_0, \cdots, H_n$ of $\mathbf{P}^n$ such that $H_0 \cap \cdots \cap H_n = \emptyset$, show that homogeneous coordinates $x_0, \dots, x_n$ can be chosen on $\mathbf{P}^n$ such that each H_i is defined by $x_i = 0$.

7. Let W be an n-dimensional vector space over k. Denote by $\mathbf{P}(W)$ the projective space $(W \setminus \{0\})/\sim$, where the equivalence relation is the usual rescaling. Show that the set of hyperplanes in $\mathbf{P}(W)$ is parametrized by $\mathbf{P}(W^*)$, where W^* is the dual vector space to W. If $P_1, \cdots, P_N$ are points of $\mathbf{P}(W)$, describe the set in $\mathbf{P}(W^*)$ corresponding to hyperplanes not containing any of the P_i. Deduce (using k infinite) that there are infinitely many such hyperplanes.

8. Let V be a hypersurface in $\mathbf{P}^n$ defined by a non-constant homogeneous polynomial F, and L a (projective) line in $\mathbf{P}^n$, i.e., a subvariety of $\mathbf{P}^n$ defined by $n-1$ linearly independent homogeneous linear equations. Show that V and L must intersect in a non-empty set.

9. Decompose the algebraic set V in $\mathbf{P}^3$ defined by equations $x_2^2 = x_1 x_3$, $x_0 x_3^2 = x_2^3$ into irreducible components.

10. Assume $\operatorname{char} k \neq 2$.

    i) Show that a homogeneous polynomial $F(x_0, x_1, x_2)$ of degree 2 can be written uniquely in the form $\mathbf{x}^T A \mathbf{x}$, where A is a 3×3 symmetric matrix with entries in k and $\mathbf{x}^T = (x_0, x_1, x_2)$; show that the polynomial is irreducible if and only if $\det(A) \neq 0$. Let $V \subset \mathbf{P}^2$ be the algebraic set defined by the equation $F = 0$, and assume F is irreducible and k algebraically closed. Show that you can choose coordinates such that $F = x_0^2 + x_1^2 + x_2^2$, and that V is isomorphic to $\mathbf{P}^1$.

    ii) In contrast, show that if $f(x, y) \in k[x, y]$ is an irreducible (non-homogeneous!) polynomial of degree 2, k algebraically closed, then $Z(f)$ is isomorphic to either $\mathbf{A}^1$ or $\mathbf{A}^1 \setminus \{0\}$.

11. Consider the projective plane curves corresponding to the following affine curves in $\mathbf{A}^2$.

(a) $y = x^3$	(b) $xy = x^6 + y^6$
(c) $x^3 = y^2 + x^4 + y^4$	(d) $x^2 y + x y^2 = x^4 + y^4$
(e) $2x^2 y^2 = y^2 + x^2$	(f) $y^2 = f(x)$ with f a polynomial of degree n.

    In each case, calculate the points at infinity of these curves, i.e., homogenize the equations to obtain equations for a curve in $\mathbf{P}^2$ and identify the resulting points at infinity. Furthermore, find the singular points of the affine curve. If you wish, you may make assumptions about the characteristic of k to simplify the analysis.

12. If $F(x_0, \dots, x_n)$ is an irreducible homogeneous polynomial of degree $d > 0$, prove that $dF = \sum_{i=0}^{n} x_i\, \partial F/\partial x_i$. If F is irreducible, let $X = Z(F) \subset \mathbf{P}^n$ be the projective variety defined by $F = 0$. In lecture, we defined the notion of $p \in X$ being a non-singlar point of X if $p \in U$ is a non-singular point, for U an affine open neighbourhood of p in X. Using the standard open affine cover $\{U_i = \mathbf{P}^n \setminus Z(x_i)\}$ of $\mathbf{P}^n$, show that the the singular locus of X (the set of points of X which are not non-singular) consists precisely of the points p in $\mathbf{P}^n$ with $\partial F/\partial x_i(p) = 0$ for $i = 0, \dots, n$. [Note: dF is $(\deg F) \cdot F$, not the differential of F!]