ALGEBRAIC GEOMETRY, SHEET I: LENT 2022

Topological Spaces of Varieties

1. Describe all the open sets in the topological subspace $\mathbb{V}(XY) \subset \mathbb{A}^2$ in the Zariski topology.
2. Equip $\mathbb{A}^1 \times \mathbb{A}^1$ with the product topology, where each factor is given the Zariski topology. Characterize all the closed sets in this topology.
3. Prove that $\mathbb{A}^2$ with the Zariski topology has the property that every open cover has a finite subcover. (If you have trouble with this, start with $\mathbb{A}^1$).
4. Let V and W be affine varieties and let $V \to W$ be a morphism. Verify that it is continuous in the Zariski topology. Suppose $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ is a set theoretic map that is continuous in the Zariski topology. Is it necessarily true that φ is a morphism?
5. Identify $\mathbb{A}^{n^2}$ with the set of complex $n \times n$ matrices. Prove that the subset $GL(n, \mathbb{C})$ of invertible matrices is Zariski dense. Prove that the set of matrices with n distinct eigenvalues is also Zariski dense.

Irreducible Components

6. Prove that in a unique factorization domain, every irreducible element is prime. Using this, prove that if f is an irreducible polynomial in $\mathbb{C}[\underline{X}]$ then $\mathbb{V}(f)$ is irreducible. Consider the variety
$$V = \mathbb{V}(X^2 - YZ) \subset \mathbb{A}^3.$$
Find an element in $\mathbb{C}[V]$ that is irreducible but not prime.
7. Let $V \subset \mathbb{A}^n_k$ be an affine variety. Suppose $V = V_1 \cup \cdots \cup V_n$ and $V = V_1' \cup \cdots \cup V_m'$ are two decompositions into irreducible varieties. Assume that no V_i is contained in V_j for $i \neq j$ and similarly for the V_i' – i.e. the decompositions are non-redundant. Prove that $n = m$ and the two decompositions coincide up to reordering.
8. Consider the ideal
$$I = \langle X^2 + Y^2 + Z^2, X^2 - Y^2 - Z^2 + 1 \rangle$$
in $\mathbb{C}[X, Y, Z]$. Let V be the variety $\mathbb{V}(I)$. Calculate the irreducible components of V.

Morphisms of varieties

9. Prove that the affine curve V given by $\mathbb{V}(XY - 1)$ in $\mathbb{A}^2$ is not isomorphic to $\mathbb{A}^1$. Calculate all morphisms
$$\mathbb{A}^1 \to V$$

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10. Let V and W be affine varieties in $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively. Prove that the product $V \times W \subset \mathbb{A}^{n+m}$ is also an affine variety. Prove that the projection
$$V \times W \to V$$
is a morphism. (More difficult: the product of irreducible varieties is also irreducible).
11. Let $\pi : \mathbb{A}^3 \to \mathbb{A}^1$ be the projection onto the first coordinate. For each point z in $\mathbb{A}^1$, the set theoretic preimage $\pi^{-1}(z)$ is isomorphic to $\mathbb{A}^2$. Denote this preimage by $\mathbb{A}^2_z$. Construct a variety $V \subset \mathbb{A}^3$ and a morphism
$$\pi : \mathbb{A}^3 \to \mathbb{A}^1$$
with the property that if $z \neq 0$, then $\pi^{-1}(z) \cap V$ is a union of two intersecting lines in $\mathbb{A}^2_z$, but $\pi^{-1}(0) \cap V$ is a union of two parallel lines in $\mathbb{A}^2_0$.

12. ($\star$) Let $V \subset \mathbb{A}^2$ be the union of the X-axis, Y-axis, and diagonal line $X = Y$. Calculate generators for $I(V)$. Let $W \subset \mathbb{A}^3$ be the union of the X, Y and Z axes. Calculate generators for $I(W)$. Show that V is not isomorphic to W.[1]

Local Geometry. The final question is meant to give you some intuition, but is to be considered non-examinable.

13. Consider the polynomial $F = Y^2 - X^2(X + 1)$ and sketch the set of real solutions to the equation $F = 0$ in $\mathbb{R}^2$. Call this set C. You may use a computer program to do this for you.

Let $\mathbb{D}_\epsilon$ be an open ball around $(0, 0)$ in $\mathbb{R}^2$ in the standard Euclidean topology of some small radius ϵ. Observe that $\mathbb{D}_\epsilon \cap C$ is homeomorphic to a union of two axes in $\mathbb{R}^2$.

Consider the ring $\mathbb{C}[\![X, Y]\!]$ of formal power series in two variables. Prove that there exists an element $G(X, Y)$ in this ring such that $G(X, Y)^2 = (1 + X)$. Deduce that the element $Y^2 - X^2(X + 1)$ can be factorized as:
$$Y^2 - X^2(X + 1) = (Y - XG)(Y + XG).$$
Note that G is an invertible element in this ring and meditate on the relationship between the two parts of this problem.

[1] One path is as follows. If p is a point on V, let $\mathfrak{m}_p$ be the set of elements in the coordinate ring that vanish at p. The quotient $\mathfrak{m}_p/\mathfrak{m}_p^2$ is a $\mathbb{C}$-vector space. What are the possible dimensions of this vector space for different choices of point p on V? How about on W?