ALGEBRAIC GEOMETRY, SHEET IV: LENT 2021

Throughout this sheet, the symbol k will denote an algebraically closed field. Curves will be smooth, projective, and irreducible unless explicitly stated otherwise.

1. Let X be a smooth degree d plane curve. Construct a morphism from X to $\mathbb{P}^1$ of degree $d-1$. Deduce that every smooth genus 1 curve admits a degree 2 morphism to $\mathbb{P}^1$. Prove that a genus 1 curve cannot admit a degree 1 morphism to $\mathbb{P}^1$.
2. Let X be a curve and $p \in X$. Prove that there exists a non-constant rational function on X that is regular away from p.
3. Let $f(x)$ be a cubic polynomial with distinct roots and assume that k has characteristic different from 2. Let X be projective closure of the affine curve $y^2 - f(x) = 0$. Consider the differential $\omega = \frac{dx}{y}$ and calculate the orders of zeros and poles for this differential at all points of X.
4. Complete the proof of the degree-genus formula outlined in Lecture 21 as follows. Let $F(X,Y,Z)$ be a homogeneous polynomial of degree d whose vanishing is a smooth plane curve C. Dehomogenise F to obtain a curve $\mathbb{V}(f(x,y))$ on an affine patch with coordinates x, y. Prove that the divisor on C associated to the differential
$$\omega = \frac{dx}{f_y}$$
is equivalent to $(d-3)H$ where H is the divisor associated to a line.
5. Let F be a bihomogeneous polynomial of bidegree (d_1, d_2) in in 4 variables Z_0, Z_1 and W_0, W_1. Assume that $\mathbb{V}(F) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth curve C. By adapting the previous exercise, calculate the degree of the canonical divisor of C and deduce that the genus of C is $(d_1-1)(d_2-1)$.
6. A curve is said to be *hyperelliptic* if it has genus at least 2 and admits a non-constant degree 2 morphism to $\mathbb{P}^1$. Using Riemann–Roch, prove that every curve of genus 2 is hyperelliptic. By using the previous exercise or otherwise, prove that there exists a hyperelliptic curve of genus g for any $g \geq 2$.
7. Prove that a smooth plane quartic curve is not hyperelliptic.
8. Let Q_1 and Q_2 be two smooth quadric surfaces in $\mathbb{P}^3$. Assume that their intersection $Q_1 \cap Q_2$ is a smooth curve. Calculate the genus of this curve. Let S be a smooth cubic surface in $\mathbb{P}^3$ and H a hyperplane. Assume that $P \cap H$ is a smooth curve and calculate its genus.
9. Prove that if X and Y are smooth projective curves then $k(X) \cong k(Y)$ if and only if $X = Y$.

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10. Let X be the projective closure of the affine curve $y^3 = x^4 + 1$. Prove that this curve is smooth and prove that it has a unique point at infinity. Calculate the zeroes and poles of the differential $\frac{dx}{y^2}$.
11. Give an example of a smooth quartic plane curve $X \subset \mathbb{P}^2$ and a line $\ell \subset \mathbb{P}^2$ such that ℓ is tangent to X at two distinct points[1]. What does this tell you about the divisor on X associated to ℓ?
12. Construct a smooth projective variety of dimension 2, i.e. an algebraic surface, that does not contain a smooth curve of genus 0. Be sure to check the smoothness and projectivity of your construction. For every non-negative h, can you construct a surface that contains no smooth curves of genus h?

[1]There is another famous geometry here: there are exactly 28 such bitangent lines to this quartic curve. Can you find them all in your example? These 28 lines are close to related to the 27 lines on the cubic surface, which is an even more famous geometry.