Mathematical Tripos: Part IB **SMP/Lent 2013**

Statistics: Example Sheet 1 (of 3)

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1. Ask your supervisor to test you on the sheet of common distributions handed out in lectures.

2. **(Probability review)** If $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$ are independent, derive the distribution of $\min(X, Y)$. If $X \sim \text{Gamma}(\alpha, \lambda)$ and $Y \sim \text{Gamma}(\beta, \lambda)$ are independent, derive the distributions of $X + Y$ and $X/(X + Y)$.

3. Suppose that $X_1, \ldots, X_n$ are iid with probability density function $f_X(x; \theta)$, and suppose that T is such that $(\mathbf{X}, T)$ has a joint probability density function. Prove the factorisation criterion for the sufficiency of T.

4. In a genetics experiment, a sample of n individuals was found to include a, b, c of the three possible genotypes GG, Gg, gg respectively. The population frequency of a gene of type G is $\theta/(\theta + 1)$, where θ is unknown, and it is assumed that the individuals are unrelated and that two genes in a single individual are independent. Show that the likelihood of θ is proportional to $\theta^{2a+b} / (1 + \theta)^{2a+2b+2c}$ and that the maximum likelihood estimate of θ is $(2a + b)/(b + 2c)$.

5. (a) Let $X_1, \ldots, X_n$ be independent Poisson random variables, with X_i having mean $i\theta$, for some $\theta > 0$. Show that $T = T(\mathbf{X}) = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ and write down the distribution of T. Show that the maximum likelihood estimator $\hat{\theta}$ of θ is a function of T, and show that it is unbiased.

   (b) For some $n > 2$, let $X_1, \ldots, X_n$ be iid with $X_i \sim \text{Exponential}(\theta)$. Find a minimal sufficient statistic T and write down its distribution. Show that the maximum likelihood estimator $\hat{\theta}$ of θ is a function of T, and show that it is biased, but asymptotically unbiased. Find an injective function h on $(0, \infty)$ such that, writing $\psi = h(\theta)$, the maximum likelihood estimator $\hat{\psi}$ of the new parameter ψ is unbiased.

6. Suppose $X_1, \ldots, X_n$ are independent random variables with distribution $\text{Bin}(1, p)$.

   (a) Show that a sufficient statistic for $\theta = (1 - p)^2$ is $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$ and that the MLE for θ is $\left(1 - \frac{1}{n}T\right)^2$.
   Hint: use the chain rule, $df/d\theta = (df/dp)(dp/d\theta)$.

   (b) Show that the MLE is a biased estimator for θ. Let $\tilde{\theta} = 1(X_1 + X_2 = 0)$. Show that $\tilde{\theta}$ is unbiased for θ. Use the Rao–Blackwell theorem to find a function of T which is an unbiased estimator for θ.

7. For some $n \geq 2$, suppose that $X_1, \ldots, X_n$ are iid random variables uniformly distributed on $[\theta, 2\theta]$ for some $\theta > 0$. Show that $\tilde{\theta} = \frac{2}{3}X_1$ is an unbiased estimator of θ. Show that $T(\mathbf{X}) = (\min_i X_i, \max_i X_i)$ is a minimal sufficient statistic for θ. Use the Rao–Blackwell theorem to find an unbiased estimator $\hat{\theta}$ of θ which is a function of T and whose variance is strictly smaller than the variance of $\tilde{\theta}$ for all $\theta > 0$.

8. (a) Let $X_1, \ldots X_n$ be iid with $X_i \sim U[0, \theta]$. Find the maximum likelihood estimator $\hat{\theta}$ of θ. Show that the distribution of $R(\mathbf{X}, \theta) = \hat{\theta}/\theta$ does not depend on θ, and use $R(\mathbf{X}, \theta)$ to find a $100(1 - \alpha)\%$ confidence interval for θ for $0 < \alpha < 1$.

   (b) The lengths (in minutes) of calls to a call centre may be modelled as iid exponentially distributed random variables, and n such call lengths are observed. The original sample is lost, but the data manager has noted down n and t where t is the total length of the n calls in minutes. Derive a 95% confidence interval for the probability that a call is longer than 2 minutes if $n = 50$ and $t = 105.3$.

9. Suppose that $X_1 \sim N(\theta_1, 1)$ and $X_2 \sim N(\theta_2, 1)$ independently, where θ_1 and θ_2 are unknown. Show that $(\theta_1 - X_1)^2 + (\theta_2 - X_2)^2$ has a χ^2_2 distribution and that this is the same as Exponential($\frac{1}{2}$), i.e., the exponential distribution with mean 2.

   Show that both the square S and circle C in $\mathbb{R}^2$, given by

$$S = \{(\theta_1, \theta_2) : |\theta_1 - X_1| \leq 2.236;\ |\theta_2 - X_2| \leq 2.236\}$$
$$C = \left\{(\theta_1, \theta_2) : (\theta_1 - X_1)^2 + (\theta_2 - X_2)^2 \leq 5.991\right\}$$

   are 95% confidence regions for (θ_1, θ_2).

   Hint: $\Phi(2.236) = (1 + \sqrt{.95})/2$, *where* Φ *is the distribution function of* $N(0, 1)$. What might be a sensible criterion for choosing between S and C?

10. Suppose that the number of defects on a roll of magnetic recording tape is modelled with a Poisson distribution for which the mean λ is known to be either 1 or 1.5. Suppose the prior mass function for λ is

$$\pi_\lambda(1) = 0.4, \quad \pi_\lambda(1.5) = 0.6\ .$$

   A random sample of five rolls of tape has $\mathbf{x} = (3, 1, 4, 6, 2)$ defects respectively. Show that the posterior distribution for λ given $\mathbf{x}$ is

$$\pi_{\lambda|\mathbf{X}}(1 \mid \mathbf{x}) = 0.012, \quad \pi_{\lambda|\mathbf{X}}(1.5 \mid \mathbf{x}) = 0.988\,.$$

11. Suppose $X_1, \ldots, X_n$ are iid with (conditional) probability density function $f(x \mid \theta) = \theta x^{\theta-1}$ for $0 < x < 1$ (and is zero otherwise), for some $\theta > 0$. Suppose that the prior for θ is Gamma(α, β), $\alpha > 0, \beta > 0$. Find the posterior distribution of θ given $\mathbf{X} = (X_1, \ldots, X_n)$ and the Bayesian estimator of θ under quadratic loss.

+12 For some $n \geq 3$, let $\epsilon_1, \ldots, \epsilon_n$ be iid with $\epsilon_i \sim N(0, 1)$. Set $X_1 = \epsilon_1$ and $X_i = \theta X_{i-1} + (1 - \theta^2)^{1/2}\epsilon_i$ for $i = 2, \ldots, n$ and some $\theta \in (-1, 1)$. Find a sufficient statistic for θ that takes values in a subset of $\mathbb{R}^3$.