STATISTICS Part IB

Example Sheet 2 (of 3) RJS/Lent 2010

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1. Let X have density function

$$f(x;\theta) = \frac{\theta}{(x+\theta)^2}, \quad x > 0,$$

where $\theta \in (0, \infty)$ is an unknown parameter. Find the likelihood ratio test of size 0.05 of $H_0 : \theta = 1$ against $H_1 : \theta = 2$ and show that the probability of Type II error is 19/21.

2. Let $X_1, \ldots X_n, Y_1, \ldots, Y_n$ be independent, with $X_1, \ldots, X_n \sim \mathrm{Exp}(\theta_1)$ and $Y_1, \ldots, Y_n \sim \mathrm{Exp}(\theta_2)$. Find the likelihood ratio test of size α of $H_0 : \theta_1 = \theta_2$ against $H_1 : \theta_1 \neq \theta_2$, expressing it in terms of the statistic

$$T = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i + \sum_{i=1}^n Y_i},$$

and the quantiles of a standard distribution.

3. Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{Exp}(\theta)$. Find the likelihood ratio test of size α of $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ $(> \theta_0)$ and write down an expression for the power function. Is the test uniformly most powerful for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$?

Find chi-squared distribution tables either by looking in a textbook or, for example, from `http://www.statsoft.com/textbook/sttable.html#chi`, or (better still) by downloading the statistical programming language **R** from `http://cran.r-project.org/` and using one of the manuals on that site to help you write your own tables. Use these tables to find the smallest sample size yielding a test of size 0.05 of $H_0 : \theta \leq 1$ against $H_1 : \theta > 1$ whose power function is at least 0.9 at $\theta = 3$.

4. A machine produces plastic articles in bunches of three articles at a time. The process is rather unreliable, and quite a few defective articles are produced. In an experimental run of the machine, 512 bunches were produced. Of these, the numbers of bunches with $i = 0, 1, 2, 3$ defective articles were 213 $(i = 0)$, 228 $(i = 1)$, 57 $(i = 2)$ and 14 $(i = 3)$. Test the hypothesis that each article has a constant (but unknown) probability θ of being defective, independently of all other articles.

5. A random sample of 59 people from the planet Krypton yielded the results below.
Carry out a chi-squared test at the 5% level that sex and eye-colour are independent factors on Krypton. Now carry out a chi-squared test at the 5% level of the null hypothesis that each of the cell probabilities is equal to 1/4. Comment on your results.

		Eye-colour	
		1 (Blue)	2 (Brown)
Sex	1 (Male)	19	10
	2 (Female)	9	21

6. Write down the model and hypotheses for a test of homogeneity in a two-way contingency table. Show that the likelihood ratio and Pearson's chi-squared tests are identical to those for the independence test. Perform the homogeneity test on the data below from a clinical trial for a drug, obtained by randomly allocating 150 patients to three equal groups.

	Improved	No difference	Worse
Placebo	18	17	15
Half dose	20	10	20
Full dose	25	13	12

7. Let $C = \{x : t(x) < k\}$ denote the critical region of a test of a simple null hypothesis based on a test statistic $T = t(X)$. Show that if the null distribution function F of T is continuous then under this hypothesis the p-value $P = F(T)$ has a $U(0,1)$ distribution.

[*Hint: For* $u \in (0,1]$, *let* $F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}$ *and show that* $x \geq F^{-1}(u)$ *iff* $F(x) \geq u$.]

8. Let f_0 and f_1 be probability mass functions on a countable set $\mathcal{X}$. State and prove a version of the Neyman–Pearson lemma for a size α test of $H_0 : f = f_0$ against $H_1 : f = f_1$ assuming that α is such that there exists a likelihood ratio test of exact size α.

9. If $X \sim N(0,1)$ and $Y \sim \chi^2_n$ are independent, we say that $T = \frac{X}{\sqrt{Y/n}}$ has a t-distribution with n degrees of freedom and write $T \sim t_n$. Derive the probability density function of T.

10. Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, where σ^2 is unknown, and suppose we are interested in testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. Derive the likelihood ratio test of size α.

11. Statisticians A and B obtain independent samples $X_1, \ldots, X_{10}$ and $Y_1, \ldots, Y_{17}$, both independent and identically distributed from a $N(\mu, \sigma^2)$ distribution with both μ and σ unknown. They estimate (μ, σ^2) by $(\bar{X}, S_{XX}/9)$ and $(\bar{Y}, S_{YY}/16)$ respectively, where, for example, $\bar{X} = \frac{1}{10}\sum_{i=1}^{10} X_i$ and $S_{XX} = \sum_{i=1}^{10}(X_i - \bar{X})^2$. Given that $\bar{X} = 5.5$ and $\bar{Y} = 5.8$, which statistician's estimate of σ^2 is more probable to have exceeded the true value by more than 50%? Find this probability (approximately) in each case.

12.* In **8.**, does there exist a version of the Neyman–Pearson lemma when a likelihood ratio test of exact size α does not exist?