Optimisation send comments to `m.tehranchi@statslab.cam.ac.uk`
Example sheet 2 - Easter 2020

1. Consider the problem

$$P : \text{minimise } \ 2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5 \ \text{ subject to } \quad \begin{aligned} x_1 + x_2 + 2x_3 + x_4 + 3x_5 &\geq 4 \\ 2x_1 - 2x_2 + 3x_3 + x_4 + x_5 &\geq 3 \\ x_1,\, x_2,\, x_3,\, x_4,\, x_5 &\geq 0. \end{aligned}$$

Write down the dual problem, and solve this graphically. Hence deduce the optimal solution to the primal problem.

The next four questions refer to Question 12 on example sheet 1.

2. Use the simplex algorithm to solve the linear program in Question 12 on example sheet 1. Let

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Choose $B = \{1, 2, 5\}$ and write $Ax = b$ in the form $A_B x_B + A_N x_N = b$ where $x_B = (x_1, x_2, x_5)^\top$, $x_N = (x_3, x_4)^\top$ and the matrices A_B and A_N are constructed appropriately.

Now write $c^\top x = c_B^\top x_B + c_N^\top x_N$ and hence write $c^\top x$ in terms of the matrices A_B, A_N and the variables x_N (i.e., eliminate x_B).

Compute A_B^{-1} and hence calculate the basic solution having B as basis. Write $c^\top x$ in terms of the non-basic variables. Prove directly from the formula for $c^\top x$ that the basic solution that you have computed is optimal for the problem maximise $c^\top x$ subject to $Ax = b$, $x \geq 0$.

Compare your answer to your answer to Question 12 on example sheet 1 and confirm that the final tableau had rows corresponding to the equation $x_B + A_B^{-1} A_N x_N = A_B^{-1} b$.

3. Given a vector $\varepsilon \in \mathbb{R}^3$, consider the linear programming problem

$$P_\varepsilon : \text{maximise } x_1 + x_2 \text{ subject to } \quad \begin{aligned} 2x_1 + x_2 &\leq 4 + \varepsilon_1 \\ x_1 + 2x_2 &\leq 4 + \varepsilon_2 \\ x_1 - x_2 &\leq 1 + \varepsilon_3 \\ x_1,\, x_2 &\geq 0 \end{aligned}$$

Find a formula in terms of ε for the optimal value for P_ε when $\|\varepsilon\|$ is very small. For what ranges of values for ε_1, ε_2, ε_3 does your formula hold?

4. Consider the linear program in Question 12 on example sheet 1, but now add the constraint $x_1 + 3x_2 \leq 6$. Apply the simplex algorithm putting x_2 into the basis at the first stage. Show that the solution at $x_1 = 0$, $x_2 = 2$ is degenerate. Try each of the possibilities for the variable leaving the basis. Explain, with a diagram, what happens.

5. For $\varepsilon > 0$, let $x_\varepsilon = (x_{1,\varepsilon}, x_{2,\varepsilon})^\top$ be the optimal solution of the problem to

$$\text{maximise } x_1 + x_2 + \varepsilon\big(\log(4 - 2x_1 - x_2) + \log(4 - x_1 - 2x_2) + \log(1 - x_1 + x_2) + \log(x_1) + \log(x_2)\big)$$
$$\text{subject to } 2x_1 + x_2 < 4,\ x_1 + 2x_2 < 4,\ x_1 - x_2 < 1,\ x_1, x_2 > 0$$

Show that $\frac{8}{3} - 5\varepsilon \le x_{1,\varepsilon} + x_{2,\varepsilon} \le \frac{8}{3}$.

6. Apply the simplex algorithm to

$$P: \text{maximise} \quad x_1 + 3x_2 \ \text{ subject to } \quad \begin{array}{rcl} x_1 - 2x_2 & \le & 4 \\ -x_1 + \ \ x_2 & \le & 3 \\ x_1,\, x_2 & \ge & 0. \end{array}$$

Explain what happens with the help of a diagram.

7. Two players fight a paint-gun duel: they face each other $2n-1$ paces apart and each has a single bullet in his gun. At a signal each may fire. If either is hit or if both fire the game ends; otherwise, both advance one pace and may again fire. The probability of either hitting his opponent if he fires after the ith pace forward ($i = 0, 1, \ldots, n-1$) is $(i+1)/n$. If a player survives after his opponent has been hit his payoff is $+1$ and his opponent's payoff is -1. The payoff is 0 if neither or both are hit. The guns are silent so that neither knows whether or not his opponent has fired. Show that, if $n = 4$, the strategy 'shoot after taking one step' is optimal for both, but that if $n = 5$ a mixed strategy is optimal. [*Hint*: $\left(0, \frac{5}{11}, \frac{5}{11}, 0, \frac{1}{11}\right)$.]

8. By considering the payoff matrix

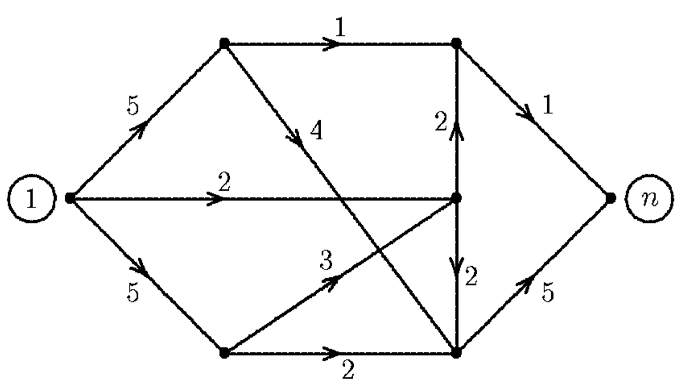
$$A = \begin{pmatrix} 0 & -2 & 3 & 0 \\ 2 & 0 & 0 & -3 \\ -3 & 0 & 0 & 4 \\ 0 & 3 & -4 & 0 \end{pmatrix}$$

show that optimal strategies for a two-person zero-sum game are not necessarily unique. Find all the optimal strategies.

9. Find optimal strategies for both players and the value of the game which has payoff matrix

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

10. Find a maximal flow and a minimal cut for the network pictured with a source at node 1 and a sink at node n.

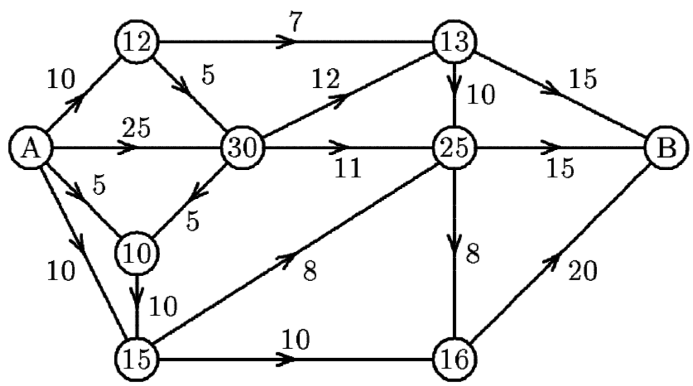


11. Devise rules for a version of the Ford–Fulkerson algorithm which works with undirected arcs.

12. How would you augment a directed network to incorporate restrictions on node capacity (the total flow permitted through a node) in maximal-flow problems?

The road network between two towns A and B pictured below. Each road is marked with an arrow giving the direction of the flow, and a number which represents its capacity. Each of the nodes of the graph represents a village. The total flow into a village cannot exceed its capacity (the number in the circle at the node). Obtain the maximal flow from A to B.

The Minister of Transport intends to build a by-pass around one of the villages, whose effect would be to completely remove the capacity constraint for that village. Which village should receive the by-pass if the intention is to increase the maximal flow from A to B by as much as possible? What would the new maximal flow be?



13. Consider a network with $2n+2$ nodes labelled $s, a_1, \ldots, a_n, b_1, \ldots, b_n, t$. Node s is the source, and node t is the sink. For each $i = 1, \ldots, n$, there is an edge (s, a_i) of capacity 1 from the source s to node a_i. For each $j = 1, \ldots, n$, there is an edge (b_j, t) of capacity 1 from node b_j to the sink t. All the other edges of the network are of the form (a_i, b_j) for some $i, j = 1, \ldots, n$ and have infinite capacity. Finally, suppose that for ever subset $A \subseteq \{a_1, \ldots, a_n\}$ the number of nodes b_j such that there exists an edge (a_i, b_j) for some $a_i \in A$ is greater than or equal to $|A|$. Prove that the maximal flow has value n. (This is, essentially, Hall's marriage theorem.)

14. Sources 1, 2, 3 stock candy floss in amounts of 20, 42, 19 tons respectively. The demand for candy floss at destinations 1, 2, 3 are 39, 34, 7 tons respectively. The matrix of transport costs per ton is

$$\begin{pmatrix} 7 & 4 & 9 \\ 8 & 12 & 5 \\ 3 & 11 & 7 \end{pmatrix}$$

with the (i, j) entry corresponding to the route $i \to j$. Find the optimal transportation scheme and the minimal cost by applying the transportation algorithm starting from (a) an assignment given by the NW method, and (b) an assignment given by the greedy algorithm.