Optimisation send comments to `m.tehranchi@statslab.cam.ac.uk`
Example sheet 1 - Easter 2020

1. Show that the set $\{x \in \mathbb{R}^n : Ax \le b, x \ge 0\}$ is convex.

2. Let $X \subseteq \mathbb{R}^n$ be convex. Show that the function $f : X \to \mathbb{R}$ is convex if and only if the set $\{(x, y) \in X \times \mathbb{R} : f(x) \le y\}$ is convex.

3. Suppose the n functions $f_1, \ldots, f_n : X \to \mathbb{R}$ are convex. Show that the function $g(x) = \max\{f_1(x), \ldots, f_n(x)\}$ is also convex.

4. Suppose that the functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are convex. For $b \in \mathbb{R}$ let

$$\varphi(b) = \inf\{f(x) : g(x) \le b, \quad x \in X\}$$

Assuming that $\varphi(b) > -\infty$ for all $b \in \mathbb{R}$, show that φ is convex.

5. Suppose $f : X \to \mathbb{R}$ is twice-differentiable and such that $D^2 f(x)$ is positive definite for all $x \in X$. Show that f is strictly convex. Find an example where $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex, but where $D^2 f(x)$ is *not* positive definite for all $x \in \mathbb{R}^n$. [Consider the case $n = 1$.]

6. Let $f(x) = x^2 + \cos x$ for all $x \in \mathbb{R}$, and consider the problem of minimising f. (a) Show that the unique minimiser is $x^* = 0$.
(b) Find constants $0 < m < M$ and $L > 0$ such that $m \le f''(x) \le M$ and $|f'''(x)| \le L$ for all x.
(c) Apply the gradient descent algorithm with step-size $t = 1/M$ and Newton's method as described in lectures. Starting with $x_0 = 1$, calculate x_1 and x_2 numberically in both cases. What do you observe?
(d) Repeat part (c) with $x_0 = 2\pi$. What do you observe now?

7. Given constants $p > 1$ and $a_1, \ldots, a_n$ and $b \ge 0$, consider the problem

$$P : \text{maximise} \sum_{i=1}^{n} a_i x_i \text{ subject to } \sum_{i=1}^{n} |x_i|^p \le b.$$

(a) Use the Lagrangian method to solve the problem.
(b) Find the value function $\varphi(b)$ of this problem. Show that φ is concave
(c) Find the dual problem D, and show that it has the same value as P.

8. Given constants $b_1,\ b_2$ such that $b_1 - e^{-b_2} \ge 0$ use the Lagrangian method to

$$\text{maximise } 2\tan^{-1} x_1 + x_2 \text{ subject to } x_1 + x_2 \le b_1,\ -\ln x_2 \le b_2,\ x_1 \ge 0,\ x_2 \ge 0.$$

[*Hint*: There will be two cases to check depending the constants b_1 and b_2.]

9. Given a $m \times n$ matrix A and a vector $b \in \mathbb{R}^n$, prove that x_0 is an extreme point of the set $\{x \in \mathbb{R}^n :\ Ax \le b,\ x \ge 0\}$ if and only if $\binom{x_0}{z_0}$ is an extreme point of the set

$$\left\{ \begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^{n+m} :\ Ax + z = b,\ x \ge 0,\ z \ge 0 \right\}.$$

where $z_0 = b - Ax_0$.

10. Consider the following problems:
(a) maximise $c^\top x$ subject to $Ax = b$;
(b) maximise $c^\top x$ subject to $Ax \leq b$;
(c) maximise $c^\top x$ subject to $Ax = b$, $x \geq 0$;
(d) maximise $c^\top x$ subject to $Ax \leq b$, $x \geq 0$.

In each case
(i) find the set Λ of values for the Lagrange multipliers λ for which the Lagrangian has a finite maximum
(ii) for each value of $\lambda \in \Lambda$ calculate the maximum of the Lagrangian and write down the dual problem;
(iii) write down the necessary and sufficient conditions for optimality;
(iv) verify that the dual of the dual is the primal problem.

11. Suppose that a linear programming problem is written in the two equivalent forms

$$\text{maximise} \quad c^\top x \quad \text{subject to} \quad Ax \leq b,\ x \geq 0,$$

where A is an $m \times n$ matrix, $c,\ x \in \mathbb{R}^n$; and

$$\text{maximise} \quad c_e^\top x_e \quad \text{subject to} \quad A_e x_e = b,\ x_e \geq 0,$$

where, after the addition of slack variables and the extension of the matrix A and vector c in the appropriate way, A_e is $m \times (n+m)$, and $c_e,\ x_e \in \mathbb{R}^{n+m}$. Use your answers to the previous question to write down the dual problem to both versions of the problem and show that the dual problems are equivalent to each other.

12. Consider the (primal) linear programming problem

$$P : \text{maximise } x_1 + x_2 \text{ subject to } \begin{array}{rcl} 2x_1 + x_2 & \leq & 4 \\ x_1 + 2x_2 & \leq & 4 \\ x_1 - x_2 & \leq & 1 \\ x_1,\ x_2 & \geq & 0 \end{array}$$

(i) Solve P graphically in the x_1-x_2 plane.
(ii) Introduce slack variables x_3, x_4 x_5 and write the problem in equality form. How many basic solutions of the constraints are there? Determine the values of $x = (x_1, \ldots, x_5)^\top$ and of the objective function at each of the basic solutions. Which of the basic solutions are feasible? Are all the basic solutions non-degenerate?
(iii) Write down the dual problem in inequality form with variables λ_1, λ_2 and λ_3; add slack variables λ_4 and λ_5 and determine the values of $\lambda = (\lambda_1, \ldots, \lambda_5)^\top$ and of the dual objective function at each of the basic solutions to the dual. Which of these are feasible for the dual?
(iv) Show that for each basic solution x to the problem P there is exactly one basic solution λ to the dual giving the same values of the primal and dual objective functions and satisfying complementary slackness ($\lambda_i x_{i+2} = 0$, $i = 1, 2, 3$ and $x_j \lambda_{j+3} = 0$, $j = 1, 2$). For how many of these matched pairs (x, λ) is x feasible for the primal problem and λ feasible for the dual?

13. * Show that a symmetric $n \times n$ matrix A is non-negative definite if and only if all eigenvalues of A are non-negative. [Recall from linear algebra that there is an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ such that each v_i is an eigenvector of A.]