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Example sheet 2 - Easter 2008

16. Consider the three equations in six unknowns given by $Ax = b$ where

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}.$$

Choose $B = \{1, 3, 6\}$ and write $Ax = b$ in the form $A_B x_B + A_N x_N = b$ where $x_B = (x_1, x_3, x_6)^{\mathrm{T}}$, $x_N = (x_2, x_4, x_5)^{\mathrm{T}}$ and the matrices A_B and A_N are constructed appropriately.

Now write $c^{\mathrm{T}} x = c_B^{\mathrm{T}} x_B + c_N^{\mathrm{T}} x_N$ and hence write $c^{\mathrm{T}} x$ in terms of the matrices A_B, A_N and the variables x_N (i.e., eliminate x_B).

Compute A_B^{-1} and hence calculate the basic solution having B as basis. For $c = (3, 1, 3, 0, 0, 0)^{\mathrm{T}}$ write $c^{\mathrm{T}} x$ in terms of the non-basic variables. Prove directly from the formula for $c^{\mathrm{T}} x$ that the basic solution that you have computed is optimal for the problem maximize $c^{\mathrm{T}} x$ subject to $Ax = b$, $x \geq 0$.

Compare your answer to your answer to Question 11 and confirm that the final tableau had rows corresponding to the equation $x_B + A_B^{-1} A_N x_N = A_B^{-1} b$. Why is it not fair to say that the simplex algorithm is just a complicated way to invert a matrix?

17. Consider the problem in Question 10 and add the constraint $x_1 + 3x_2 \leq 6$. Apply the simplex algorithm putting x_2 into the basis at the first stage. Show that the solution at $x_1 = 0$, $x_2 = 2$ is degenerate. Try each of the possibilities for the variable leaving the basis. Explain, with a diagram, what happens.

18. In the previous example the additional constraint was redundant (it did not change the feasible set). Can degeneracy occur without redundant equations?

19. Show that introducing slack variables in a linear programming problem does not change the extreme points of the feasible set by proving that x is an extreme point of the set $\{x : Ax \leq b, x \geq 0\}$ if and only if $\binom{x}{z}$ is an extreme point of the set

$$\left\{ \binom{x}{z} : Ax + z = b, \ x \geq 0, \ z \geq 0 \right\}.$$

20. Give sufficient conditions for strategies p and q to be optimal for a two-person zero-sum game with payoff matrix A and value v.

Two players fight a duel: they face each other $2n - 1$ paces apart and each has a single bullet in his gun. At a signal each may fire. If either is hit or if both fire the game ends; otherwise, both advance one pace and may again fire. The probability of either hitting his opponent if he fires after the ith pace forward $(i = 0, 1, \ldots, n - 1)$ is $(i + 1)/n$. If a player survives after his opponent has been hit his payoff is $+1$ and his opponent's payoff is -1. The payoff is 0 if neither or both are hit. The guns are silent so that neither knows whether or not his opponent has fired. Show that, if $n = 4$, the strategy 'shoot after taking one step' is optimal for both, but that if $n = 5$ a mixed strategy is optimal. [*Hint*: $\left(0, \frac{5}{11}, \frac{5}{11}, 0, \frac{1}{11}\right)$.]

21. By considering the payoff matrix

$$A = \begin{pmatrix} 0 & -2 & 3 & 0 \\ 2 & 0 & 0 & -3 \\ -3 & 0 & 0 & 4 \\ 0 & 3 & -4 & 0 \end{pmatrix}$$

show that optimal strategies for a two-person zero-sum game are not necessarily unique. Find all the optimal strategies.

22. The $n \times n$ matrix of a two-person zero-sum game is such that the row and column sums all equal s. Show that the game has value s/n. [*Hint*: Guess a solution and show that it is optimal.]

23. Find optimal strategies for both players and the value of the game which has payoff matrix

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

[You may like to try to compare the effort required to solve this by

(1) seeking strategies and a value which satisfy the optimality conditions;
(2) direct solution of Player I's original minimax problem; and
(3) using the simplex method on one of the player's problems after transforming it as suggested in lectures.]

24. Find a maximal flow and a minimal cut for the network pictured with a source at node 1 and a sink at node n.
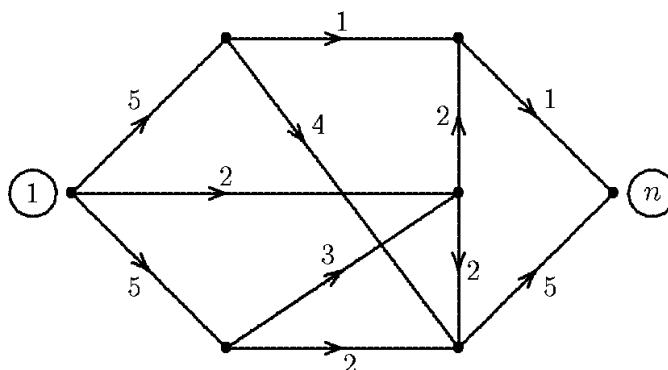
25. Devise rules for a version of the Ford–Fulkerson algorithm which works with undirected arcs.

As a consequence of a drought, an emergency water supply must be pumped from Oxbridge to Camford along the network of pipes pictured below. The numbers against the pipes show their maximal capacities, and each pipe may be used in either direction. Find the maximal flow and prove that it is maximal.
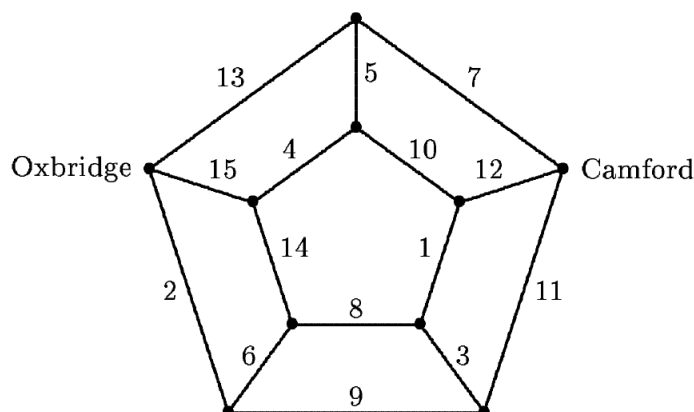
26. How would you augment a directed network to incorporate restrictions on node capacity (the total flow permitted through a node) in maximal-flow problems?
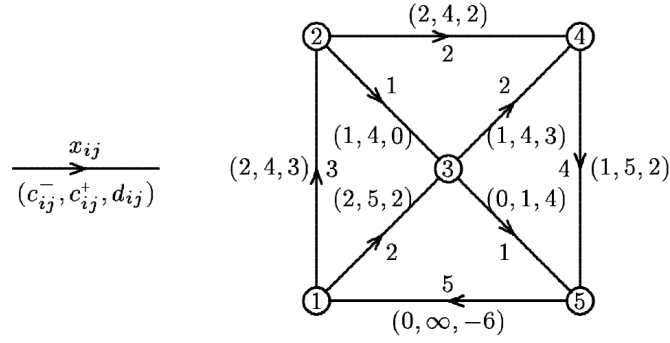
The road network between two towns A and B pictured below. Each road is marked with an arrow giving the direction of the flow, and a number which represents its capacity. Each of the nodes of the graph represents a village. The total flow into a village cannot exceed its capacity (the number in the circle at the node). Obtain the maximal flow from A to B.

The Minister of Transport intends to build a by-pass around one of the villages, whose effect would be to completely remove the capacity constraint for that village. Which village should receive the by-pass if the intention is to increase the maximal flow from A to B by as much as possible? What would the new maximal flow be?
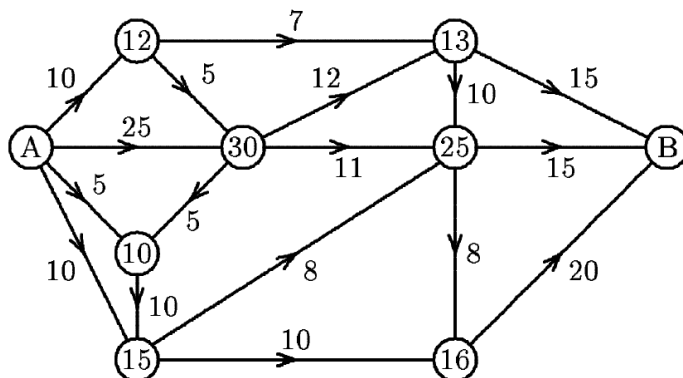
27. By finding a suitable potential on the nodes of the network in the figure (i.e., a set of suitable node numbers), show that the flow illustrated below is a minimal-cost circulation. [Each arc is labelled with the flow, x_{ij}, and with a triple of numbers giving the constants $(c_{ij}^-, c_{ij}^+, d_{ij})$ for that arc.]

28. Consider the problem of assigning lecturers $L_1, \ldots, L_n$ to courses $C_1, \ldots, C_n$ so as to minimize student dissatisfaction. The dissatisfaction felt by students if lecturer L_i gives course C_j is d_{ij}, and each lecturer must give exactly one course. Show how to formulate this problem as a problem of minimizing the cost of a circulation in a network. (Can you be sure that your network problem has an optimal flow of the appropriate kind?)

$$\xrightarrow{\quad x_{ij} \quad}$$
$$(c^-_{ij}, c^+_{ij}, d_{ij})$$

Network diagram labels:

(2,4,2) on edge 2 → 4, with 2

②   ④

1   2

(1,4,0)   (1,4,3)

(2,4,3) 3   ③   4 (1,5,2)

(2,5,2)   (0,1,4)

2   1

①   5   ⑤

(0, ∞, −6)

For the example of 3 lecturers and 3 courses with dissatisfaction matrix

$$\begin{pmatrix} 6 & 3 & 1 \\ 8 & 12 & 5 \\ 3 & 11 & 7 \end{pmatrix}$$

find an optimal flow through the appropriate network (by guessing) and compute node numbers for each node so that the optimality conditions are satisfied.

29. Sources 1, 2, 3 stock candy floss in amounts of 20, 42, 19 tons respectively. The demand for candy floss at destinations 1, 2, 3 are 39, 34, 7 tons respectively. The matrix of transport costs per ton is

$$\begin{pmatrix} 7 & 4 & 9 \\ 8 & 12 & 5 \\ 3 & 11 & 7 \end{pmatrix}$$

with the (i, j) entry corresponding to the route $i \to j$. Find the optimal transportation scheme and the minimal cost by applying the transportation algorithm starting from (a) an assignment given by the NW method, and (b) an assignment given by the greedy algorithm.