Optimization Michael Tehranchi/DPK

Example sheet 1 - Easter 2008

1. Minimize each of the following functions in the region specified.

(i) $3x$ in $\{x : x \geq 0\}$; (ii) $x^2 - 2x + 3$ in $\{x : x \geq 0\}$; (iii) $x^2 + 2x + 3$ in $\{x : x \geq 0\}$.

For each of the following functions specify the set Λ of λ values for which the function has a finite minimum in the region specified, and for each $\lambda \in \Lambda$ find the minimum value and (all) optimal x.

(iv) λx subject to $x \geq 0$; (v) λx subject to $x \in \mathbb{R}$; (vi) $\lambda_1 x^2 + \lambda_2 x$ subject to $x \in \mathbb{R}$; (vii) $\lambda_1 x^2 + \lambda_2 x$ subject to $x \geq 0$; (viii) $(\lambda_1 - \lambda_2)x$ subject to $0 \leq x \leq M$.

2. Minimize $x^{\mathrm{T}} V x$ subject to $\mu^{\mathrm{T}} x = m$ where V is a symmetric, positive definite $n \times n$ matrix, and $\mu \in \mathbb{R}^n$ is a fixed vector.

[Suppose an investor can choose from among n stock. If

- x_i = the number of shares of stock i the investor holds,
- μ_i = the mean return on stock i, and
- V_{ij} = the covariance between the returns of stock i and stock j,

then the problem amounts to minimizing the variance of the portfolio subject to a given mean return m. Markowitz was awarded the Nobel Prize in Economics in 1990 in part for his analysis of this problem.]

3. Maximize $n_1 \log p_1 + \ldots + n_k \log p_k$ subject to $p_1 + \ldots + p_k = 1$, $p_1, \ldots, p_k > 0$, where $n_1, \ldots, n_k$ are positive constants. [The optimal $(p_1, \ldots, p_k)$ is the maximum likelihood estimator for the multinomial distribution.]

4. For a probability measure $\mathbb{P}$ on a finite set S, the entropy of $\mathbb{P}$ is defined as

$$H(\mathbb{P}) = -\sum_{i \in S} p_i \log p_i,$$

where $p_i = \mathbb{P}\{i\}$ and $0 \log 0 = 0$ by convention. Find the maximum and the minimum values of $H(\mathbb{P})$.

5. Maximize $2 \tan^{-1} x_1 + x_2$ subject to $x_1 + x_2 \leq b_1$, $-\ln x_2 \leq b_2$, $x_1 \geq 0$, $x_2 \geq 0$, where b_1, b_2 are constants such that $b_1 - e^{-b_2} \geq 0$.

6. Let S be a finite sample space, and let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on S. Consider the problem

$$\text{maximize } \mathbb{Q}(A) \text{ subject to } \mathbb{P}(A) \leq \alpha.$$

What is the dual problem? [This problem is related to the Neyman–Pearson lemma in statistics.]

Hint: Let $p_i = \mathbb{P}\{i\}$ and $q_i = \mathbb{Q}\{i\}$ for each $i \in S$. Then the problem can be reformulated as

$$\text{maximize } \sum_{i \in S} q_i x_i \text{ subject to } \sum_{i \in S} p_i x_i \leq \alpha, \quad x_i \in \{0, 1\}$$

7. Show that the quadratic programs

$$P : \text{minimize } \frac{1}{2}x^{\mathrm{T}}Qx + c^{\mathrm{T}}x \text{ subject to } x \geq b$$

and

$$D : \text{maximize } -\frac{1}{2}y^{\mathrm{T}}Q^{-1}y - b^{\mathrm{T}}y + b^{\mathrm{T}}c \text{ subject to } y \leq c$$

are dual problems, where Q is a symmetric, positive definite $n \times n$ matrix and $b, c \in \mathbb{R}^n$.

8. Consider the following problems:
(a) minimize $c^{\mathrm{T}}x$ subject to $Ax = b$;
(b) minimize $c^{\mathrm{T}}x$ subject to $Ax \leq b$;
(c) minimize $c^{\mathrm{T}}x$ subject to $Ax = b$, $x \geq 0$;
(d) minimize $c^{\mathrm{T}}x$ subject to $Ax \leq b$, $x \geq 0$.

In each case
(i) find the set Λ of values for the Lagrange multipliers λ for which the Lagrangian has a finite minimum (subject to the appropriate regional constraint, if any);
(ii) for each value of $\lambda \in \Lambda$ calculate the minimum of the Lagrangian and write down the dual problem;
(iii) write down the necessary and sufficient conditions for optimality;
(iv) verify that the dual of the dual is the primal problem.

9. Suppose that a linear programming problem is written in the two equivalent forms

$$\text{minimize} \quad c^{\mathrm{T}}x \quad \text{subject to} \quad Ax \leq b, \ x \geq 0,$$

where A is an $m \times n$ matrix, c, $x \in \mathbb{R}^n$; and

$$\text{minimize} \quad c_e^{\mathrm{T}}x_e \quad \text{subject to} \quad A_e x_e = b, \ x_e \geq 0,$$

where, after the addition of slack variables and the extension of the matrix A and vector c in the appropriate way, A_e is $m \times (n+m)$, and c_e, $x_e \in \mathbb{R}^{n+m}$. Use your answers to the previous question to write down the dual problem to both versions of the problem and show that the dual problems are equivalent to each other.

10. Consider the (primal) linear programming problem

$$P : \text{maximize } x_1 + x_2 \text{ subject to } \begin{aligned} 2x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 4 \\ x_1 - x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(i) Solve P graphically in the x_1-x_2 plane.
(ii) Introduce slack variables x_3, x_4 x_5 and write the problem in equality form. How many basic solutions of the constraints are there? Determine the values of $x = (x_1, \ldots, x_5)^{\mathrm{T}}$ and of the objective function at each of the basic solutions. Which of the basic solutions are feasible? Are all the basic solutions non-degenerate?

(iii) Write down the dual problem in inequality form with variables λ_1, λ_2 and λ_3; add slack variables λ_4 and λ_5 and determine the values of $\lambda = (\lambda_1, \ldots, \lambda_5)^{\mathrm{T}}$ and of the dual objective function at each of the basic solutions to the dual. Which of these are feasible for the dual?

(iv) Show that for each basic solution x to the problem P there is exactly one basic solution λ to the dual giving the same values of the primal and dual objective functions and satisfying complementary slackness ($\lambda_i x_{i+2} = 0$, $i = 1, 2, 3$ and $x_j \lambda_{j+3} = 0$, $j = 1, 2$). For how many of these matched pairs (x, λ) is x feasible for the primal problem and λ feasible for the dual?

(v) Solve the problem P using the simplex algorithm starting with the initial basic feasible solution $x_1 = x_2 = 0$. Try both choices of the variable to introduce into the basis on the first step. Compare the objective rows of the various tableaux generated with appropriate basic solutions of the dual problem. What do you observe?

11. Use the simplex algorithm to solve

$$P: \text{maximize} \quad 3x_1 + x_2 + 3x_3 \quad \text{subject to} \quad \begin{aligned} 2x_1 + x_2 + x_3 &\le 2 \\ x_1 + 2x_2 + 3x_3 &\le 5 \\ 2x_1 + 2x_2 + x_3 &\le 6 \\ x_1,\, x_2,\, x_3 &\ge 0, \end{aligned}$$

Each row of the final tableau is the sum of scalar multiples of the rows of the initial tableau. Explain how to determine the scalar multipliers directly from the final tableau.

Let $P(\epsilon)$ be the linear programming problem when the right-hand side $b = (2, 5, 6)^{\mathrm{T}}$ is replaced by the perturbed vector $b(\epsilon) = (2 + \epsilon_1, 5 + \epsilon_2, 6 + \epsilon_3)^{\mathrm{T}}$. Give a formula, in terms of $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)^{\mathrm{T}}$, for the optimal value for $P(\epsilon)$ when the ϵ_i are small. For what ranges of values for ϵ_1, ϵ_2, ϵ_3 does your formula hold?

12. Apply the simplex algorithm to

$$P: \text{maximize} \quad x_1 + 3x_2 \quad \text{subject to} \quad \begin{aligned} x_1 - 2x_2 &\le 4 \\ -x_1 + x_2 &\le 3 \\ x_1,\, x_2 &\ge 0. \end{aligned}$$

Explain what happens with the help of a diagram.

13. Use the two-phase algorithm to solve:

$$\text{maximize} \quad -2x_1 - 2x_2 \quad \text{subject to} \quad \begin{aligned} 2x_1 - 2x_2 &\le 1 \\ 5x_1 + 3x_2 &\ge 3 \\ x_1,\, x_2 &\ge 0. \end{aligned}$$

[*Hint*: You should get $x_1 = \frac{9}{16}$, $x_2 = \frac{1}{16}$. Note that it is possible to choose the first pivot column so that Phase I lasts only one step, but this requires a different choice of pivot column than the one specified by the usual rule-of-thumb.]

14. Use the two-phase algorithm to solve:

$$\text{minimize} \quad 13x_1 + 5x_2 - 12x_3 \text{ subject to} \quad \begin{aligned} 2x_1 + x_2 + 2x_3 &\leq 5 \\ 3x_1 + 3x_2 + x_3 &\geq 7 \\ x_1 + 5x_2 + 4x_3 &= 10 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

15. Consider the problem

$$\text{minimize} \quad 2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5 \quad \text{subject to} \quad \begin{aligned} x_1 + x_2 + 2x_3 + x_4 + 3x_5 &\geq 4 \\ 2x_1 - 2x_2 + 3x_3 + x_4 + x_5 &\geq 3 \\ x_1,\ x_2,\ x_3,\ x_4,\ x_5 &\geq 0. \end{aligned}$$

Write down the dual problem, and solve this graphically. Hence deduce the optimal solution to the primal problem.