METRIC AND TOPOLOGICAL SPACES, SHEET I: 2019

1. Describe all the topologies on the set $\{0, 1\}$ and specify which ones yield connected topological spaces. Compute the closure of the subset $\{0\}$ in each of these topologies.
2. Describe the open sets in $(\mathbb{R}, \tau_{\text{Eucl}})$ that are also closed. Justify your answer.
3. Consider the following subsets $\mathbb{R}^2$ and decide if they are open, closed, or neither.
   - $A_1 = \{(x, y) : x < 0\}$
   - The subset A_2 of points (x, y) in $\mathbb{R}^2$ that are lexicographically strictly larger than $(0, 0)$. (That is, if $x > 0$ then y can be anything, but if $x = 0$, then $y > 0$.)
   - $A_3 = \{(x, y) : y \in \mathbb{Q}, \;\; \& \;\; y = x^n \;\; \text{for some positive integer } n\} \subset \mathbb{R}^2$.
4. Let $\mathbf{C}[0, 1]$ be the set of continuous functions on $[0, 1]$. We define metrics d_1 and d_∞ as follows. For functions f and g, define
$$d_1(f, g) = \int_{[0,1]} |f(t) - g(t)| \, dt, \qquad d_\infty(f, g) = \max_{t \in [0,1]} |f(t) - g(t)|.$$
   - Prove that these are both metrics on the set $\mathbf{C}[0, 1]$.
   - Prove that these metrics are not equivalent.
5. On the vector space $\mathbb{R}^n$ define the following two metrics. For points $x = (x_1, \ldots x_n)$ and $y = (y_1, \ldots, y_n)$ define:
$$d_1(x, y) = \sum_i |x_i - y_i|, \qquad d_\infty(x, y) = \max_i |x_i - y_i|.$$
   Prove that these are both equivalent metrics and they are both equivalent to the usual metric. Sketch the unit ball around the origin in $\mathbb{R}^2$ in each of these metrics.
6. Let X be a topological space. If X is Hausdorff, then show that
$$\Delta X = \{(x, x) : x \in X\} \subset X \times X$$
   is closed.
7. Fix a prime number p. The *p-adic norm* on the rational numbers $\mathbb{Q}$ is defined as follows. Write a nonzero rational number z as
$$z = p^n \frac{a}{b},$$
   where a and b are integers not divisible by p, and n is an integer. Then define
$$|z|_p = p^{-n}.$$
   The *p-adic metric* is defined by
$$d_p(x, y) = |x - y|_p.$$

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Prove the *ultrametric triangle inequality*:

$$|x+y|_p \leq \max(|x|_p, |y|_p).$$

Deduce the triangle inequality for d_p from this.

8. Prove or give a counterexample:
   - A non-constant continuous map $f : X \to Y$ is an *open* map, i.e. the image of every open set in X is an open set in Y.
   - If $f : X \to Y$ is continuous and bijective, then it is a homeomorphism.
   - If $f : X \to Y$ is continuous, open, and bijective, then it must be a homeomorphism.
9. A continuous map $X \to Y$ is a *closed map* if the image of every closed set is closed. Give an example to show that an open continuous map need not be closed.
10. Show that the quotient space

    $$[0,1] \cup [2,3]/1 \sim 2$$

    is homeomorphic to the closed interval $[0,1]$.
11. Let X and Y be metric spaces. Show that $X \times Y$ admits a metric whose induced topology is the product topology on $X \times Y$. In other words, show that products of metrizable topological spaces are metrizable.
12. Let G be a topological group. That is, G is a set with a topology, a distinguished identity element $e \in G$, and a group operation, such that the multiplication map $m : G \times G \to G$ and the inverse map $i : G \to G$ are continuous and satisfy the usual group axioms.
    - Given two points $x, y \in G$, show that there exists a homeomorphism $G \to G$ taking x to y.
    - If $\{e\}$ is a closed subset, then show that the diagonal $\Delta G \subset G \times G$ is a closed subgroup.
    - If $\{e\}$ is a closed subset, then show that the center

      $$Z(G) = \{g \in G : gh = hg, \quad \text{for all } h \in G\}$$

      is a closed normal subgroup of G.
    - Let H be a normal subgroup of G. The set of cosets G/H inherits a quotient topology from G. Prove that this is an open map.
13. Prove that the Zariski topology on $\mathbb{C}^2$ is not the same as the product of Zariski topologies on $\mathbb{C} \times \mathbb{C}$.
14. Let X be a topological space and Y be a Hausdorff topological space. Let $f, g : X \to Y$ be two continuous maps. Prove that the set

    $$W = \{x \in X : f(x) = g(x)\}$$

    is closed in X.