Metric and Topological Spaces **Easter 2016**

Example Sheet 2

1. Which of the following subsets of $\mathbb{R}^2$ are a) connected b) path connected?
   (a) $B_1((1,0)) \cup B_1((-1,0))$
   (b) $\overline{B}_1((1,0)) \cup B_1((-1,0))$
   (c) $\{(x,y) \mid y = 0 \text{ or } x/y \in \mathbb{Q}\}$
   (d) $\{(x,y) \mid y = 0 \text{ or } x/y \in \mathbb{Q}\} - \{(0,0)\}$

2. Suppose that X is connected, and that $f : X \to Y$ is a locally constant map; *i.e.* for every $x \in X$, there is an open neighborhood U of x such that $f(y) = f(x)$ for all $y \in U$. Show that f is constant.

3. Show that the product of two connected spaces is connected.

4. Show there is no continuous injective map $f : \mathbb{R}^2 \to \mathbb{R}$.

5. Show that $\mathbb{R}^2$ with the topology induced by the British rail metric is not homeomorphic to $\mathbb{R}^2$ with the topology induced by the Euclidean metric.

6. Let X be a topological space. If A is a connected subspace of X, show that $\overline{A}$ is also connected. Deduce that any component of X is a closed subset of X.

7. (a) If $f : [0,1] \to [0,1]$ is continuous, show there is some $x \in [0,1]$ with $f(x) = x$.
   (b) Suppose $f : [0,1] \to \mathbb{R}$ is continuous and has $f(0) = f(1)$. For each integer $n > 1$, show that there is some $x \in [0,1]$ with $f(x) = f(x + \frac{1}{n})$.

8. Is there an infinite compact subset of $\mathbb{Q}$?

9. If $A \subset \mathbb{R}^n$ is not compact, show there is a continuous function $f : A \to \mathbb{R}$ which is not bounded.

10. If X is a topological space, its *one point compactification* X^+ is defined as follows. As a set, X^+ is the union of X with an additional point ∞. A subset $U \subset X^+$ is open if either
    (a) $\infty \notin U$ and U is an open subset of X
    (b) $\infty \in U$ and $X^+ - U$ is a compact, closed subset of X.

    Show that X^+ is a compact topological space. If $X = \mathbb{R}^n$, show that $X^+ \simeq S^n$.

11. Suppose that X is a compact Hausdorff space, and that C_1 and C_2 are disjoint closed subsets of X. Show that there exist open subsets $U_1, U_2 \subset X$ such that $C_i \subset U_i$ and $U_1 \cap U_2 = \emptyset$.

12. Let (X, d) be a metric space. A complete metric space (X', d') is said to be a *completion* of (X, d) if a) $X \subset X'$ and $d'|_{X \times X} = d$ and b) X is dense in X'.

    (a) Suppose that (Y, d_Y) is a complete metric space and that $f : X \to Y$ is an *isometric embedding, i.e.* $d_Y(f(x_1), f(x_2)) = d(x_1, x_2)$. Show that f extends to an isometric embedding $f' : X' \to Y$.

    (b) Deduce that any two completions of X are *isometric, i.e.* related by an bijective isometric embedding.

13. If p is a prime number, let $\mathbb{Z}_p$ be the space of sequences $(x_n)_{n \geq 0}$ in $\mathbb{Z}/p\mathbb{Z}$, equipped with the metric $d((x_n), (y_n)) = p^{-k}$, where k is the smallest value of n such that $x_n \neq y_n$.

    (a) Find an isometric embedding of $f : (\mathbb{Z}, d_p) \to \mathbb{Z}_p$, where d_p is the p-adic metric. Show that $\mathbb{Z}_p$ is a completion of the image of f. The set $\mathbb{Z}_p$ is called the p-adic numbers.

    (b) Show that $\mathbb{Z}_p$ is compact and totally disconnected.

    (c) Show that the maps $f, g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $f(x, y) = x + y$, $g(x, y) = xy$ extend to continuous maps $f', g' : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$.

    (d) Let a be an integer which is relatively prime to p and assume $p > 2$. Show that the equation $x^2 = a$ has a solution in $\mathbb{Z}_p$ if and only if it has a solution in $\mathbb{Z}/p\mathbb{Z}$.

14. Show that $C[0, 1]$ equipped with the uniform metric is complete.

15. Define a norm $\|\cdot\|_{\infty,\infty}$ on $C^1[0, 1]$ by $\|f\|_{\infty,\infty} = \max\{\|f\|_\infty, \|f'\|_\infty\}$. Let $B = \overline{B}_1(0)$ be the closed unit ball in this norm. Show that any sequence (f_n) in B has a subsequence which converges with respect to the sup norm. (Hint: first find a subsequence (f_{n_i}) such that $f_{n_i}(x)$ converges for all $x \in \mathbb{Q} \cap [0, 1]$.) Deduce that the closure of B in $(C[0, 1], d_\infty)$ is compact.

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