Part 1B Metric and Topological Spaces 2012 pmhw@dpmms.cam.ac.uk

Example Sheet 2

(1) Which of the following subspaces of $\mathbf{R}^2$ are (a) connected, (b) path-connected? Here $B((x,y),\delta)$, respectively $\bar{B}((x,y),\delta)$, represents the open, respectively closed, balls of radius δ centred on (x,y).
(i) $B((1,0),1) \cup B((-1,0),1)$; (ii) $B((1,0),1) \cup \bar{B}((-1,0),1)$;
(iii) $\{(x,y) \ : \ x = 0 \text{ or } y/x \in \mathbf{Q}\}$; (iv) $\{(x,y) \ : \ x = 0 \text{ or } y/x \in \mathbf{Q}\} \setminus \{(0,0)\}$.

(2) Suppose that X is a connected topological space, Y is any topological space and $f : X \to Y$ a locally constant map, i.e. for every $x \in X$, there is an open neighbourhood U on which f is constant. Show that f is constant.

(3) Let $A \subset \mathbf{R}^2$ be a set of points satisfying the following two conditions:
(a) if $x \in \mathbf{Q}$, then $(x,y) \in A$ for every $y \in \mathbf{R}$;
(b) for every $x \in \mathbf{R}$, there is at least one $y \in \mathbf{R}$ for which $(x,y) \in A$.
Show that A is connected.

(4) Which of the following subspaces of $\mathbf{R}$ are homeomorphic: $(a,b], (a,b), [a,\infty), [a,b]$, $(-\infty,\infty)$, where $a < b$ in $\mathbf{R}$?

(5) Show that there is no continuous injective map from $\mathbf{R}^2$ to $\mathbf{R}$.

(6) Consider the real line $\mathbf{R}$ with the *half-open interval topology*, with a base of open sets given by $[a,b)$ with $a < b$ in $\mathbf{R}$; show that this space is also *totally disconnected*, i.e. the only connected subsets are single points. Show also that the closed interval $[a,b]$, where $a < b$ in $\mathbf{R}$, is not compact.

(7) If p is a prime, show that the rationals $(\mathbf{Q}, d_p)$ with the p-adic topology are totally disconnected.

(8) Let $A = \{(x, \sin(1/x)) \ : \ x > 0\} \cup \{(0,y) \ : \ -1 \le y \le 1\}$ in $\mathbf{R}^2$; prove that A is connected but not path-connected.

(9) A family of sets in a topological space is said to have the *finite intersection property* if every finite subfamily has a non-empty intersection. Prove that a space X is compact if and only if, for every family of closed subsets $\{V_a\}_{a\in A}$ of X with the finite intersection property, the whole family has non-empty intersection.

(10) Let X be a topological space. The *one-point compactification* X^+ of X is setwise $X \cup \{\infty\}$, for an additional point which we denote by ∞, with topology given by $U \subset X^+$ is open if *either* $U \subset X$ is open in X, *or* $U \ni \infty$ and $X^+ \setminus U$ is both closed and compact

in X. Prove that X^+ is a topological space, and that it is compact. When $X = \mathbf{C}$, show that X^+ is homeomorphic to the sphere $S^2 \subset \mathbf{R}^3$.

(11) Let X be $C[0,1]$ endowed with the sup metric. Show that the closed unit ball $\bar{B}(0,1)$, where 0 denotes the zero-function, is not compact. Is X a connected topological space?

(12) Let $\bar{D} \subset \mathbf{C}$ denote the closed unit disc with boundary $C = \{z \ : \ |z| = 1\}$. Show that the space $\bar{D}/C$ obtained by identifying all points of C to a single point in the quotient, is homeomorphic to $S^2 \subset \mathbf{R}^3$.

(13) Consider the two dimensional torus $X = \mathbf{R}^2/\sim$, where $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 - x_2$ and $y_1 - y_2$ are both integers. Let $Y \subset \mathbf{R}^3$ be the embedded torus (with subspace topology) given by points

$$((2 + \cos\theta)\cos\phi, (2 + \cos\theta)\sin\phi, \sin\theta),$$

for $0 \leq \theta < 2\pi, 0 \leq \phi < 2\pi$. Show that both spaces are homeomorphic to $S^1 \times S^1$ with the product topology.

(14) Let S_1 be the quotient space obtained by identifying the north and south poles on S^2. Let S_2 be the quotient space given by collapsing one circle $\{\text{pt}\} \times S^1$ inside the 2-dimensional torus $S^1 \times S^1$ to a point. Prove that S_1 is homeomorphic to S_2.

(15) Show that a subset $A \subset \mathbf{R}^n$ is compact if and only if every continuous function on A is bounded.

(16) Let X be a compact Hausdorff space. Given disjoint closed sets F_1, F_2 in X, prove that there exist open subsets U_1, U_2 of X with $U_1 \cap U_2 = \emptyset$, $F_1 \subset U_1$ and $F_2 \subset U_2$. [A space with this latter property is called *normal.* Question 6 on Example Sheet 1 shows that any metric space, whether compact or otherwise, is normal.]

(17) Let C_n, $n = 1, 2, \ldots$, be compact, connected, non-empty subsets of a Hausdorff space X such that $C_1 \supset C_2 \supset C_3 \supset \ldots$. Prove that the intersection $\cap_{n=1}^{\infty} C_n$ is connected — you will need to use Q16 here. Show by example that the compactness assumption may not be omitted.

(18) If a metric space (Y, d) has a countable dense subset, prove that it has a countable base of open sets. Show moreover that if Y has a countable base of open sets, then so does any subspace Z (with the subspace topology).

*Taking $X = \mathbf{R}$ with the half-open interval topology (as defined in Question 6, prove that the product $X \times X$ has a countable dense subset. If $Z = \{(x, y) \ : \ x + y = 1\} \subset X \times X$, show that the subspace topology on Z is discrete. Deduce that $X \times X$ does not have a countable base of open sets. Deduce that $X \times X$ is not *metrizable* (i.e. its topology cannot be a metric topology), and hence that X is not metrizable.