

Metric & Topological spaces, Sheet 1: 2011

1. (a) Give a bounded open subset of $(\mathbb{R}, \mathcal{T}_{eucl})$ which is not a finite union of open intervals.
 (b) Is the trigonometric function $\sin : (\mathbb{R}, \mathcal{T}_{Zariski}) \rightarrow (\mathbb{R}, \mathcal{T}_{Zariski})$ continuous?
 (c) Give a topology on \mathbb{R} which is neither trivial nor discrete such that every open set is closed (and vice-versa).
 (d) Let \mathcal{T} be the topology on \mathbb{R} for which open sets are \emptyset , \mathbb{R} and open intervals of the form $(-\infty, a)$. Show this is a topology, and describe the closure of the singleton set $\{a\}$. What are the continuous functions from $(\mathbb{R}, \mathcal{T})$ to $(\mathbb{R}, \mathcal{T}_{eucl})$?
 (e) Give \mathbb{R} the following topology: a subset H is closed in \mathbb{R} if and only if H is closed and bounded in the usual (Euclidean metric) topology. Show that this is a topology, that points are closed sets, but that this topology is not Hausdorff.
 (f) Define a subset of the integers \mathbb{Z} to be open either if it is empty or if for some $k \in \mathbb{Z}$ the set S contains all integers $\geq k$. Show this defines a topology. Is it metrisable?
2. Which of the following are open in $(\mathbb{R}^2, \mathcal{T}_{eucl})$? [Convincing pictures are sufficient.]
 (i) $\{y > x^2\}$
 (ii) $\{y > x^2, y \leq 1\}$
 (iii) $\{y > x^2, y \leq -1\}$.
3. Prove or give counterexamples to:
 (i) A continuous function $f : X \rightarrow Y$ is an *open map* i.e. if $U \subset X$ is an open subset then $f(U)$ is an open subset of Y .
 (ii) If $f : X \rightarrow Y$ is continuous and bijective (that is, one-to-one and onto) then f is a homeomorphism.
 (iii) If $f : X \rightarrow Y$ is continuous, open and bijective then f is a homeomorphism.
4. Let X and Y be topological spaces.
 (a) If $X = A \cup B$ is a union of (not necessarily disjoint) closed subsets, prove that a function f on X is continuous if and only if $f|_A$ and $f|_B$ are continuous functions on A, B respectively, where A, B have the subspace topology induced from X . (The notation refers to the restriction of the function to the appropriate domain.)
 (b) Show $f : X \rightarrow Y$ is continuous if and only if for all $A \subset X$, $f(cl(A)) \subset cl(f(A))$. Deduce that if f is surjective, the continuous image of a dense set is dense.
 (c) If now X and Y are metric spaces, show $f : X \rightarrow Y$ is continuous if it preserves limits of sequences, i.e. if for every sequence $(x_n) \subset X$ converging to $a \in M_1$, the sequence $(f(x_n)) \subset Y$ converges to $f(a)$.
5. Let $f, g : X \rightarrow Y$ be continuous functions where X is any topological space and Y is a Hausdorff topological space. Prove that $W = \{x \in X \mid f(x) = g(x)\}$ is a closed subspace of X . Deduce that the fixed point set of a continuous function on a Hausdorff space is closed.
6. Let $X = \mathbb{Z}_{>0}$ be the set of strictly positive integers. Define a topology on X by saying that the *basic* open sets are the arithmetic progressions

$$U_{a,b} = \{na + b \in X \mid n \in \mathbb{Z}\}$$

for pairs $(a, b) \in X \times X$ with $a \neq 0$ and $hcf(a, b) = 1$. (Hence, a general open set is a union of these.)

- (a) Show this does define a topology. Is it Hausdorff ?
- (b) If p is prime, show that $U_{p,0}$ is a closed subset of X . Deduce there are infinitely many prime numbers.
7. (a) Show the quotient space $([0, 1] \cup [2, 3])/1 \sim 2$ is homeomorphic to a closed interval.
 (b) Define an equivalence relation \sim on the interval $[0, 1] \subset \mathbb{R}$ by $x \sim y \iff x - y \in \mathbb{Q}$. Describe the quotient space I/\sim .
8. Let S_1 be the quotient space given by identifying the north and south poles on the 2-sphere. Let S_2 be the quotient space given by collapsing one circle $S^1 \times \{pt\}$ inside the 2-dimensional torus $S^1 \times S^1$ to a point. Draw pictures of S_1 and S_2 and sketch an argument to show that they are homeomorphic.
9. (a) Prove that the product of two metric spaces admits a metric inducing the product topology.
 (b) Show that if the product of two metric spaces is complete, then so are the factors.
 (c) Let M denote the space of bounded sequences of real numbers with the *sup* metric. Show (i) the subspace of convergent sequences is complete and (ii) the subspace of sequences with only finitely many non-zero values is not complete.
 (d) Let M be a complete metric space and $f : M \rightarrow M$ be continuous. Suppose for some r the iterate $f^{or} = f \circ \dots \circ f$ (r times) is a contraction. Prove f has a unique fixed point.
10. Let G be a topological group; so G is a topological space and there are given a distinguished point $e \in G$, continuous functions $m : G \times G \rightarrow G$ and $i : G \rightarrow G$ (multiplication and inverse) which satisfy the (usual, algebraic) group axioms. Typical examples are matrix groups like $SL_2(\mathbb{R})$, $SO(3)$, the unit circle in \mathbb{C} , etc. Prove the following:
 (a) G is homogeneous: given any $x, y \in G$ there is a homeomorphism $\phi : G \rightarrow G$ such that $\phi(x) = y$.
 (b) If $\{e\} \subset G$ is a closed subset, then the diagonal $\Delta G = \{(g, g) \mid g \in G\} \subset G \times G$ is a closed subgroup of $G \times G$.
 (c) If $\{e\}$ is closed in G then the centre $Z(G) = \{g \in G \mid gh = hg \forall h \in G\}$ is a closed normal subgroup of G .
 (d) Let H be a subgroup of G . Give the set of cosets $(G : H)$ the quotient topology from the natural projection $\pi : G \rightarrow (G : H)$. Prove that π is an open map (cf. question 3).
 (e) Prove $(G : H)$ is Hausdorff if and only if H is closed in G .
11. (a) Show that a space X may be homeomorphic to a subspace of a space Y whilst Y is homeomorphic to a subspace of X , but where X, Y are not themselves homeomorphic.
 (b)* Give an example of a pair of spaces X, Y which are not homeomorphic but for which $X \times I$ and $Y \times I$ are homeomorphic, where I denotes the unit interval with its usual topology. [Hint: draw a flat handbag, i.e. two arcs attached to a disk.]