EXAMPLE SHEET 1

1. Give an example of a metric space X which has a closed ball of radius 1.001 which contains 100 *disjoint* closed balls of radius one.

2. Show that the sequence $2008, 20008, 200008, 2000008, \dots$ converges in the 5-adic metric.

3. Suppose that $\mathbb{R} \times \mathbb{R}$ is endowed with one of the product metrics defined in lectures (pick your favourite). Show that the map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = x + y$ is continuous.

4. Suppose that (X, d) is a metric space and that $X \times X$ is endowed with the product metric $\tilde{d}((x, x'), (y, y')) = d(x, y) + d(x', y')$. Show that the metric d, viewed as a map from $X \times X$ to $\mathbb{R}$, is continuous.

5. By a *norm* on $\mathbb{R}^n$ we mean a function $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ for which $\|x\| = 0$ if and only if $x = 0$, for which $\|x + y\| \leqslant \|x\| + \|y\|$, and such that $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}$. Show that any norm defines a metric on $\mathbb{R}^n$ by the rule $d(x, y) = \|x - y\|$. Suppose now that $(X_1, d_1), \dots, (X_n, d_n)$ are metric spaces. Wikipedia says that the function $\tilde{d}$ defined by

$$\tilde{d}((x_1, x_2, \dots, x_n), (x_1', x_2', \dots, x_n')) = \|(d_1(x_1, x_1'), \dots, d_n(x_n, x_n'))\|$$

is a metric on $X_1 \times \cdots \times X_n$. Do you believe this?[1]

6. Suppose that $(X_1, d_1), (X_2, d_2), \dots$ are metric spaces. Prove the claim made in lectures, that

$$\tilde{d}((x_1, x_2, \dots), (x_1', x_2', \dots)) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i, x_i')}{1 + d_i(x_i, x_i')}$$

defines a metric on the countably infinite product $\prod_{i=1}^{\infty} X_i$.

7. How many topologies are there on the set $\{1, 2\}$? Show that if $n > 1$ then there are at least 2^n different topologies on the set $\{1, \dots, n\}$. Is the number of topologies on $\{1, \dots, n\}$ bounded above by C^n, for some universal constant C?

8. Give an example of an infinite topological space X which is homeomorphic to $X \times X$.

9. Exhibit a countable basis for the usual topology on $\mathbb{R}$.

10. Suppose that $C[0, 1]$ is endowed with the L^2-metric $d(f, g) = \sqrt{\int_0^1 |f(x) - g(x)|^2 \, dx}$. Is it complete?

[1] Update: I am pleased to report that next year's example sheet will see "says" replaced by "said".

11. Which of the following pairs of topological spaces are homeomorphic? Justify your answers. [*Hint: I'm not sure all of this can be done using material from the first six lectures of the course.*]

(i) A coffee mug and a doughnut;
(ii) $C[0,1]$ and $C[0,2]$ (both with the metric $d(f,g) = \max|f(x) - g(x)|$);
(iii) $(0,1)$ and $[0,1]$;
(iv) $\{0,1,\dots,9\}^{\mathbb{N}}$ and $[0,1]$;
(v) $\mathbb{R}^2 \setminus \{0\}$ and $\mathbb{R}^2 \setminus \{x : |x| \leqslant 1\}$.

12. Is $\mathbb{Q}$ complete in the 2-adic metric?

13. For this question we work in $\mathbb{R}^n$ with the Euclidean metric. Show that there do not exist two disjoint closed unit balls inside any closed ball of radius 2. Show that there do, however, exist $(1+c)^n$ disjoint closed unit balls inside any closed ball of radius 3.001, for some absolute constant $c > 0$. Do there exist exponentially many disjoint closed unit balls inside a closed ball of radius 2.001?

14. Suppose that A is a set inside some topological space X. Show that no more than 14 distinct sets may be obtained from A by repeated applications of the closure and complementation operations (e.g. "the closure of the complement of the closure of the closure of the complement of the complement of A"). Show furthermore that 14 is the best possible constant in this result.

15. Show that the set $S \subseteq C[0,1]$ consisting of continuous functions which map $\mathbb{Q}$ to $\mathbb{Q}$ is dense, where the metric on $C[0,1]$ is defined by $d(f,g) = \max|f(x) - g(x)|$.

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