Metric & Topological Spaces, sheet 2: 2007

1. Which of the following subspaces of $\mathbb{R}^2$ are (a) connected (b) path-connected? $B_t(x, y)$ denotes the open t-disc about $(x, y) \in \mathbb{R}^2$ and $\overline{X} = cl(X)$ denotes closure.
(i) $B_1(1, 0) \cup B_1(-1, 0)$;
(ii) $\overline{B_1(1, 0)} \cup \overline{B_1(-1, 0)}$
(iii) $B_1(1, 0) \cup \overline{B_1(-1, 0)}$
(iv) $\{(x, y) \,|\, x = 0 \text{ or } y/x \in \mathbb{Q}\}$.

2. (a) Let $\phi : [0, 1] \to [0, 1]$ be continuous. Prove that ϕ has a fixed point.

   (b) Prove that an odd degree real polynomial has a real root.

   (c) Let $\mathbb{S}^1 \subset \mathbb{R}^2$ denote the unit circle in the Euclidean plane (with the subspace topology) and let $f : \mathbb{S}^1 \to \mathbb{R}$ be continuous. Prove there is some $x \in \mathbb{S}^1$ such that $f(x) = f(-x)$.

   (d) Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and has $f(0) = f(1)$. For each integer $n \geq 2$ show there is some x s.t. $f(x) = f(x + \frac{1}{n})$.

3. Prove there is no continuous function $f : [0, 1] \to \mathbb{R}$ such that $x \in \mathbb{Q} \Leftrightarrow f(x) \notin \mathbb{Q}$ (where $\mathbb{Q}$ denotes the rational numbers).

4. (i) Give an example of a sequence of closed connected subsets $C_n \subset \mathbb{R}^2$ s.t. $C_n \supset C_{n+1}$ but $\bigcap_{n=1}^{\infty} C_n$ not connected.

   (ii) If $C_n \subset X$ is compact and connected in a Hausdorff space, and $C_n \supset C_{n+1}$ for each n, show $\bigcap_{n=1}^{\infty} C_n$ *is* connected.

5. Suppose $A \subset \mathbb{R}^n$ is not compact. Show there is a continuous function on A which is not bounded.

6. Let X be a topological space. The *one-point compactification* X^+ of X is set-wise the union of X and an additional point ∞ (thought of as “at infinity”) with the topology: $U \subset X^+$ is open if either

   (i) $U \subset X$ is open in X or

   (ii) $U = V \cup \{\infty\}$ where $V \subset X$ and $X \backslash V$ is both compact and closed in X.

   Prove that X^+ is a topological space and prove that it is compact (N.B. regardless of whether X is compact or not!).

7. (a) Using connectedness arguments, prove that the three intervals $[0, 1], (0, 1)$ and $[0, 1)$ are pairwise not homeomorphic. Prove the same result using compactness.

   (b) Prove that the Euclidean spaces $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ are pairwise not homeomorphic.

8. (a) Draw an example of a surface in $\mathbb{R}^3$ with infinitely many “ends” (i.e. for which complements of arbitrarily large compact sets have infinitely many connected components).

   (b) Prove that a discrete space is totally disconnected (meaning, the maximal connected subsets are points). Does the converse always hold? Give an example of a compact totally disconnected subset of $\mathbb{R}$.

9. A family of sets has the *finite intersection property* if and only if every *finite* subfamily has non-empty intersection. Prove that a space X is compact if and only if whenever $\{V_a\}_{a \in A}$ is a family of closed subsets of X with the finite intersection property, the whole family has non-empty intersection.

10. Let (X, d) be a compact metric space. Prove that a subspace $Z \subset X$ is compact only if every sequence in Z has a subsequence which converges in the metric to a point of Z. [Note: the question requires "only if" and not "if".]

    Let X be the space of continuous functions from $[0, 1]$ to the reals $\mathbb{R}$ with the metric $d(f, g) = sup\{|f(x) - g(x)| \ : \ x \in [0, 1]\}$. Prove that the unit ball $\{u \in X \mid d(0, u) \leq 1\}$ is *not compact*, where 0 denotes the obvious zero-function. [Thus the "Heine-Borel" theorem is *not* valid in arbitrary metric spaces.]

11. Let M be a compact metric space and suppose that for every $n \in \mathbb{Z}_{\geq 0}$, $V_n \subset M$ is a closed subset and $V_{n+1} \subset V_n$. Prove that

$$\text{diameter}(\bigcap_{n=1}^{\infty} V_n) \ = \ \inf\{\text{diameter}(V_n) \mid n \in \mathbb{Z}_{\geq 0}\}.$$

    [Hint: suppose the LHS is smaller by some amount ϵ.]

12. (a) Let X be a compact topological space. Prove that for any topological space T the second projection map $X \times T \to T$ is a closed map (i.e. the image of any closed set is closed).

    (b) Let $f : X \to Y$ be an arbitrary function, Y be a compact space and suppose the graph $\Gamma_f \subset X \times Y$ is closed. Prove that f is continuous.

    (c) Prove that if the continuous map ϕ is proper (cf. question 11) then $\phi \times \text{id} : X \times T \to Y \times T$ is closed, for any topological space T and $\text{id} : T \to T$ the identity map.

    NOTE: *The last two questions are for the geometrically inclined to play with: they're way off syllabus. Try and get an informal idea of what's going on, rather than worrying about rigorous proofs.*

13. * (a) Let X be a topological space and $x_0 \in X$ a distinguished point. Show that the set of connected components of the based loop space $\Omega X = \{\gamma : [0, 1] \to X \,|\, \gamma(0) = \gamma(1) = x_0\}$ forms a group.

    (b) Give examples in which this group is non-trivial. Can it be non-trivial and finite? Can it be non-abelian?

    (c) Let Σ_g denote a compact surface of genus $g \geq 1$ in $\mathbb{R}^3$, that is, one with g holes. Show that the set Γ_g of homeomorphisms from Σ_g to itself naturally admits the structure of a topological space. Persuade yourself there are homeomorphisms of Σ_g which act non-trivially on the group of components of $\Omega\Sigma_g$. Deduce that Γ_g itself has infinitely many connected components. [This is not true when $g = 0$, i.e. for the sphere.]

14. * (a) Draw a knot κ, that is the image of an embedding $S^1 \to \mathbb{R}^3$. Explain why there is always a "nice" (smooth) compact embedded surface in $\mathbb{R}^3$ with boundary the knot (think of soap films!). Is there always an orientable such surface, i.e. one in which "turning clockwise" makes sense globally on the surface?

    (b) What is the smallest genus (number of holes) of such a surface with boundary the *trefoil* knot, which is the knot which lies on a torus in $\mathbb{R}^3$ and winds twice round one way and three times round the other?

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