Metric & Topological spaces, Sheet 1: 2007

1. (a) Let $(X, \mathcal{T}_X)$ be a discrete topological space and $(Y, \mathcal{T}_Y)$ an arbitrary topological space. Prove that every function $X \to Y$ is necessarily continuous.

   (b) Give a topology on $\mathbb{R}$ which is neither trivial nor discrete such that every open set is closed (and vice-versa).

   (c) Give a bounded open subset of $(\mathbb{R}, \mathcal{T}_{eucl})$ which is a not a finite union of open intervals.

   (d) Find a topological space X and a decomposition $X = Y \cup Z$ into disjoint topological subspaces Y, Z which are both dense subsets of X.

2. (a) Define a subset of the integers $\mathbb{Z}$ to be open either if it is empty or if for some $k \in \mathbb{Z}$ the set S contains all integers $\geq k$. Show this defines a topology. Is it metrisable ?

   (b) Let $\mathcal{T}$ be the topology on $\mathbb{R}$ for which open sets are ϕ, $\mathbb{R}$ and open intervals of the form $(-\infty, a)$. Show this is a topology, and describe the closure of the singleton set $\{a\}$. What are the continuous functions from $(\mathbb{R}, \mathcal{T})$ to $(\mathbb{R}, \mathcal{T}_{eucl})$?

   (c) Give $\mathbb{R}$ the following topology: a subset H is closed in $\mathbb{R}$ if and only if H is closed in the usual (Euclidean) topology and also bounded (of finite length in the usual Euclidean distance). Show that this *is* a topology, that points are closed sets, but that this topology is not Hausdorff.

3. (a) Show that a function $f : M_1 \to M_2$ of metric spaces is continuous if it preserves limits of sequences, i.e. if for every sequence $(x_n) \subset M_1$ converging to $a \in M_1$, the sequence $(f(x_n)) \subset M_2$ converges to $f(a)$.

   (b) For a function $f : X \to Y$, show f is continuous if and only if for all $A \subset X$, $f(cl(A)) \subset cl(f(A))$.

   (c) Write a topological space $X = Y \cup Z$ as a union of (not necessarily disjoint) closed subsets. Prove that a function f on X is continuous if and only if $f|_Y$ and $f|_Z$ are continuous functions on Y, Z respectively, where Y, Z have the subspace topology induced from X. (The notation refers to the restriction of the function to the appropriate domain.)

4. Prove or give counterexamples to:

   (i) A continuous function $f : X \to Y$ is an *open map* i.e. if $U \subset X$ is an open subset then $f(U)$ is an open subset of Y.

   (ii) If $f : X \to Y$ is continuous and bijective (that is, one-to-one and onto) then f is a homeomorphism.

   (iii) If $f : X \to Y$ is continuous, open and bijective then f is a homeomorphism.

5. Which of the following are open in $(\mathbb{R}^2, \mathcal{T}_{eucl})$? [Draw pictures, detailed proofs are not needed.] (i) $\{y > x^2\}$ (ii) $\{y > x^2, y \leq 1\}$, (iii) $\{y > x^2, y \leq -1\}$.

6. Show that a space X may be homeomorphic to a subspace of a space Y whilst Y is homeomorphic to a subspace of X, but where X, Y are not themselves homeomorphic.

7. (a) Prove that the product of two metric spaces admits a metric which induces the product topology.

   (b) Show that if the product of two metric spaces is complete, then so are the factors.

(c) Let M denote the space of bounded sequences of real numbers with the *sup* metric. Show (i) the subspace of convergent sequences is complete and (ii) the subspace of sequences with only finitely many non-zero values is not complete.

(d) Let M be a complete metric space and $f : M \to M$ be continuous. Suppose for some r the iterate $f^{\circ r} = f \circ \cdots \circ f$ (r times) is a contraction. Prove f has a unique fixed point in M.

8. Let $f, g : X \to Y$ be continuous functions where X is any topological space and Y is a Hausdorff topological space. Prove that $W = \{x \in X \mid f(x) = g(x)\}$ is a closed subspace of X. Deduce that the fixed point set of any continuous function on a Hausdorff space is closed.

9. Define an equivalence relation $\sim$ on the interval $[0, 1] \subset \mathbb{R}$ by $x \sim y \iff x - y \in \mathbb{Q}$. Show that the quotient space $I/\sim$ is indiscrete.

10. Let C be the cylinder
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ 0 \leq z \leq 1\}.$$
Using a picture, or otherwise, describe the quotient spaces of C by the following two equivalence relations:

(a) $(x, y, z) \sim (x', y', z') \iff x = -x',\ y = -y',\ z = z'$.

(b) $(x, y, z) \sim (x', y', z') \iff x = -x',\ y = -y',\ z = 1 - z'$.

11. * Let G be a topological group; so G is a topological space and there are given a distinguished point $e \in G$, continuous functions $m : G \times G \to G$ and $i : G \to G$ (multiplication and inverse) which satisfy the (usual, algebraic) group axioms. (Typical examples are matrix groups like $SL_2(\mathbb{R})$, the unit circle in $\mathbb{C}$, etc.) Prove the following:

(a) G is homogeneous: given any $x, y \in G$ there is a homeomorphism $\phi : G \to G$ such that $\phi(x) = y$.

(b) If $\{e\} \subset G$ is a closed subset, then the diagonal $\Delta G = \{(g, g) \mid g \in G\} \subset G \times G$ is a closed subgroup of $G \times G$.

(c) If $\{e\}$ is closed in G then the centre $Z(G) = \{g \in G \mid gh = hg\ \forall h \in G\}$ is a closed normal subgroup of G.

(d) Let H be an algebraic subgroup of G. Give the set of cosets $(G : H)$ the quotient topology from the natural projection map $\pi : G \to (G : H)$. Prove that π is an open map (images of open sets are open). [NB The usual notation for $(G : H)$ is G/H but this clashes with our notation for "collapsing subsets to a point".]

(e) Prove $(G : H)$ is Hausdorff if and only if H is closed in G.

12. * Give an example of a pair of spaces X, Y which are not homeomorphic but for which $X \times I$ and $Y \times I$ are homeomorphic, where I denotes the unit interval with its usual topology.

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