Linear Algebra: Example Sheet 3 of 4

1. Show that none of the following matrices are similar:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Is the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

similar to any of them? If so, which?

2. Find a basis with respect to which $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ is in Jordan normal form. Hence compute $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^n$.

3. (a) Recall that the Jordan normal form of a 3×3 complex matrix can be deduced from its characteristic and minimal polynomials. Give an example to show that this is not so for 4×4 complex matrices.
(b) Let A be a 5×5 complex matrix with $A^4 = A^2 \neq A$. What are the possible minimal and characteristic polynomials? How many possible JNFs are there for A? [*You probably don't want to list them all.*]

4. Let α be an endomorphism of the finite dimensional vector space V over F, with characteristic polynomial $\chi_\alpha(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_0$. Show that $\det(\alpha) = (-1)^n c_0$ and $\operatorname{tr}(\alpha) = -c_{n-1}$.

5. Let α be an endomorphism of the finite-dimensional vector space V, and assume that α is invertible. Describe the eigenvalues and the characteristic and minimal polynomial of α^{-1} in terms of those of α.

6. Prove that any square complex matrix is similar to its transpose. Now prove that that the inverse of a Jordan block $J_m(\lambda)$ with $\lambda \neq 0$ has Jordan normal form a Jordan block $J_m(\lambda^{-1})$. For an arbitrary non-singular square matrix A, describe the Jordan normal form of A^{-1} in terms of that of A.

7. Let V be a complex vector space of dimension n and let α be an endomorphism of V with $\alpha^{n-1} \neq 0$ but $\alpha^n = 0$. Show that there is a vector $\mathbf{x} \in V$ for which $\mathbf{x},\ \alpha(\mathbf{x}),\ \alpha^2(\mathbf{x}),\ \ldots,\ \alpha^{n-1}(\mathbf{x})$ is a basis for V. Give the matrix of α relative to this basis.
Let $p(t) = a_0 + a_1 t + \ldots + a_k t^k$ be a polynomial. What is the matrix for $p(\alpha)$ with respect to this basis? What is the minimal polynomial for α? What are the eigenvalues and eigenvectors?
Show that if an endomorphism β of V commutes with α then $\beta = p(\alpha)$ for some polynomial $p(t)$.
[*It may help to consider* $\beta(\mathbf{x})$.]

8. Let A be an $n \times n$ matrix all the entries of which are real. Show that the minimal polynomial of A, over the complex numbers, has real coefficients.

9. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, with $a_i \in \mathbb{C}$, and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ a_n & a_0 & a_1 & \ldots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \ldots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}.$$

Show that the determinant of C is $\det C = \prod_{j=0}^{n} f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n+1))$.

10. Let V be a 4-dimensional vector space over $\mathbb{R}$, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the basis of V^* dual to the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ for V. Determine, in terms of the ξ_i, the bases dual to each of the following:
(a) $\{\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_3\}$;
(b) $\{\mathbf{x}_1, 2\mathbf{x}_2, \frac{1}{2}\mathbf{x}_3, \mathbf{x}_4\}$;
(c) $\{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_3 + \mathbf{x}_4, \mathbf{x}_4\}$;
(d) $\{\mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 + \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_3 + \mathbf{x}_2 - \mathbf{x}_1\}$.

11. Let P_n be the space of real polynomials of degree at most n. For $x \in \mathbb{R}$ define $\varepsilon_x \in P_n^*$ by $\varepsilon_x(p) = p(x)$. Show that $\varepsilon_0, \ldots, \varepsilon_n$ form a basis for P_n^*, and identify the basis of P_n to which it is dual.

12. (a) Show that if $\mathbf{x} \neq \mathbf{y}$ are vectors in the finite dimensional vector space V, then there is a linear functional $\theta \in V^*$ such that $\theta(\mathbf{x}) \neq \theta(\mathbf{y})$.
(b) Suppose that V is finite dimensional. Let $A, B \leq V$. Prove that $A \leq B$ if and only if $A^o \geq B^o$. Show that $A = V$ if and only if $A^o = \{\mathbf{0}\}$. Deduce that a subset $F \subset V^*$ of the dual space spans V^* if and only if $\{\mathbf{v} \in V : f(\mathbf{v}) = 0 \text{ for all } f \in F\} = \{\mathbf{0}\}$.