Linear Algebra: Example Sheet 2 of 4

1. (Another proof of the row rank column rank equality.) Let A be an $m \times n$ matrix of (column) rank r. Show that r is the least integer for which A factorises as $A = BC$ with $B \in \mathrm{Mat}_{m,r}(\mathbb{F})$ and $C \in \mathrm{Mat}_{r,n}(\mathbb{F})$. Using the fact that $(BC)^T = C^T B^T$, deduce that the (column) rank of A^T equals r.

2. Write down the three types of elementary matrices and find their inverses. Show that an $n \times n$ matrix A is invertible if and only if it can be written as a product of elementary matrices. Write the following matrices as products of elementary matrices. Then use this to find their inverses.
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix}.$$

3. Let $\lambda \in F$. Evaluate the determinant of the $n \times n$ matrix A with each diagonal entry equal to λ and all other entries 1.

4. Let A be an $n \times m$ matrix. Prove that if B is an $m \times n$ matrix then $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$.

5. Let A and B be $n \times n$ matrices over a field F. Show that the $2n \times 2n$ matrix
$$C = \begin{pmatrix} I & B \\ -A & 0 \end{pmatrix} \quad \text{can be transformed into} \quad D = \begin{pmatrix} I & B \\ 0 & AB \end{pmatrix}$$
by elementary row operations (which you should specify). By considering the determinants of C and D, obtain another proof that $\det AB = \det A \det B$.

6. (i) Let V be a non-trivial real vector space of finite dimension. Show that there are no endomorphisms α, β of V with $\alpha\beta - \beta\alpha = \mathrm{id}_V$.
(ii) Let V be the space of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. Find endomorphisms α, β of V which do satisfy $\alpha\beta - \beta\alpha = \mathrm{id}_V$.

7. Compute the characteristic polynomials of the matrices
$$\begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$
Which of the matrices are diagonalisable over $\mathbb{C}$? Which over $\mathbb{R}$?

8. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

9. Let $a_0, ..., a_n$ be distinct real numbers, and let
$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{pmatrix}.$$
Show that $\det(A) \neq 0$.

10. For $n \geq 2$, let $A, B \in \mathrm{Mat}_n(\mathbb{F})$. Show that, if A and B are invertible, then

$$(i)\ \mathrm{adj}\,(AB) = \mathrm{adj}\,(B)\mathrm{adj}\,(A)\,, \quad (ii)\ \det(\mathrm{adj}\,A) = (\det A)^{n-1}\,, \quad (iii)\ \mathrm{adj}\,(\mathrm{adj}\,A) = (\det A)^{n-2}A\,.$$

Show that

$$\mathrm{rank}(\mathrm{adj}\,A) = \begin{cases} n & \text{if} \quad \mathrm{rank}(A) = n \\ 1 & \text{if} \quad \mathrm{rank}(A) = n-1 \\ 0 & \text{if} \quad \mathrm{rank}(A) \leq n-2. \end{cases}$$

Do (i)-(iii) hold if A is singular? [Hint: for (i) consider $A + \lambda I$ for $\lambda \in \mathbb{F}$.]

11. Let V be a vector space, let $\pi_1, \pi_2, \ldots, \pi_k$ be endomorphisms of V such that $\mathrm{id}_V = \pi_1 + \cdots + \pi_k$ and $\pi_i \pi_j = 0$ for any $i \neq j$. Show that $V = U_1 \oplus \cdots \oplus U_k$, where $U_j = \mathrm{Im}(\pi_j)$.
Let α be an endomorphism on the vector space V, satisfying the equation $\alpha^3 = \alpha$. Prove directly that $V = V_0 \oplus V_1 \oplus V_{-1}$, where V_λ is the λ-eigenspace of α.

12. Let $A \in \mathrm{Mat}_n(\mathbb{R})$ be such that for all $i \in \{1, \ldots, n\}$ we have $|a_{i,i}| > \sum_{j \neq i} |a_{i,j}|$. Show that A is invertible. Now additionally assume $a_{i,i} \geq 0$, for all i; does this imply $\det(A) > 0$?

13. Let $A \in \mathrm{Mat}_n(\mathbb{C})$ satisfy $A^k = I$, for some $k \in \mathbb{N}$. Show that A can be diagonalised.