Michaelmas Term 2022

Linear Algebra: Example Sheet 4 of 4

Exercices 15-16-17-18 are more difficult and optional.

1. The square matrices A and B over the field F are congruent if $B = P^T AP$ for some invertible matrix P over F. Which of the following symmetric matrices are congruent to the identity matrix over $\mathbb{R}$, and which over $\mathbb{C}$? (Which, if any, over $\mathbb{Q}$?) Try to get away with the minimum calculation.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}.$$

2. Find the rank and signature of the following quadratic forms over $\mathbb{R}$.

$$x^2 + y^2 + z^2 - 2xz - 2yz, \quad x^2 + 2y^2 - 2z^2 - 4xy - 4yz, \quad 16xy - z^2, \quad 2xy + 2yz + 2zx.$$

   If A is the matrix of the first of these (say), find a non-singular matrix P such that $P^T AP$ is diagonal with entries ± 1.

3.

4. Let A, B in $\mathcal{M}_n(\mathbb{R})$ and define $(A|B) = \mathrm{tr}(AB^T)$.
   (a) Show that $(\cdot|\cdot)$ is a scalar product on $\mathcal{M}_n(\mathbb{R})$. Deduce that $|\mathrm{tr}(AB^T)| \leq \mathrm{tr}(AA^T)^{1/2}\mathrm{tr}(BB^T)^{1/2}$.
   (b) Show that the map $A \mapsto \mathrm{tr}(A^2)$ is a quadratic form on $\mathrm{Mat}_n(\mathbb{R})$. Find its rank and signature.
   (c) Given $A \in \mathcal{M}_n(\mathbb{R})$, we let $\mathrm{ad}_A(X) = AX - XA$. Compute the adjoint of ad_A for $(\cdot|\cdot)$.
   (d) Show that the A is nilpotent $\Leftrightarrow A \in \mathrm{Im}(\mathrm{ad}_A) \Leftrightarrow$ A is similar to $2A$.

5. Let $\psi : V \times V \to \mathbb{C}$ be a Hermitian form on a complex vector space V.
   (i) Find the rank and signature of ψ in the case $V = \mathbb{C}^3$ and

$$\psi(x, x) = |x_1 + ix_2|^2 + |x_2 + ix_3|^2 + |x_3 + ix_1|^2 - |x_1 + x_2 + x_3|^2.$$

   (ii) Show in general that if $n > 2$ then $\psi(u, v) = \frac{1}{n}\sum_{k=1}^{n} \zeta^k \psi(u + \zeta^k v, u + \zeta^k v)$ where $\zeta = e^{2\pi i/n}$.

6. Show that the quadratic form $2(x^2+y^2+z^2+xy+yz+zx)$ is positive definite. *Write down* an orthonormal basis for the corresponding inner product on $\mathbb{R}^3$. Compute the basis of $\mathbb{R}^3$ obtained by applying the Gram-Schmidt process to the standard basis with respect to this inner product.

7. Let $W \leq V$ with V an inner product space. An endomorphism π of V is called an *idempotent* if $\pi^2 = \pi$. Show that the orthogonal projection onto W is a self-adjoint idempotent. Conversely show that any self-adjoint idempotent is orthogonal projection onto its image.

8. An endomorphism α of a finite dimensional inner product space V is *positive definite* if it is self-adjoint and satisfies $\langle \alpha(\mathbf{x}), \mathbf{x} \rangle > 0$ for all non-zero $\mathbf{x} \in V$.
   (i) Prove that a positive definite endomorphism has a unique positive definite square root.
   (ii) Let α be an invertible endomorphism of V and α^* its adjoint. By considering $\alpha^*\alpha$, show that α can be factored as $\beta\gamma$ with β unitary and γ positive definite.

9. Let V be a finite dimensional complex inner product space, and let α be an endomorphism on V. Assume that α is *normal*, that is, α commutes with its adjoint: $\alpha\alpha^* = \alpha^*\alpha$. Show that α and α^* have a common eigenvector $\mathbf{v}$, and the corresponding eigenvalues are complex conjugates. Show that the subspace $\langle \mathbf{v} \rangle^\perp$ is invariant under both α and α^*. Deduce that there is an orthonormal basis of eigenvectors of α.

10. Find a linear transformation which simultaneously reduces the pair of real quadratic forms

$$2x^2 + 3y^2 + 3z^2 - 2yz, \qquad x^2 + 3y^2 + 3z^2 + 6xy + 2yz - 6zx$$

to the forms

$$X^2 + Y^2 + Z^2, \qquad \lambda X^2 + \mu Y^2 + \nu Z^2$$

for some $\lambda, \mu, \nu \in \mathbb{R}$ (which should turn out in this example to be integers).

Does there exist a linear transformation which reduces the pair of real quadratic forms $x^2 - y^2, \quad 2xy$ simultaneously to diagonal forms?

11. Let P_n be the ($n+1$-dimensional) space of real polynomials of degree $\leq n$. Define

$$(f, g) = \int_{-1}^{+1} f(t)g(t)dt\,.$$

Show that (,) is an inner product on P_n and that the endomorphism $\alpha : P_n \to P_n$ defined by

$$\alpha(f)(t) = (1 - t^2)f''(t) - 2tf'(t)$$

is self-adjoint. If f is an eigenvector of α of degree k, what is the corresponding eigenvalue? Why must α have precisely one monic eigenvector of degree k for each $0 \leq k \leq n$?

Let $s_k \in P_n$ be defined by $s_k(t) = \frac{d^k}{dt^k}(1 - t^2)^k$. Prove the following.
(i) For $i \neq j$, $(s_i, s_j) = 0$ and $(s_0, \ldots, s_n)$ forms a basis for P_n.
(iii) For all $1 \leq k \leq n$, s_k spans the orthogonal complement of P_{k-1} in P_k.
(iv) s_k is an eigenvector of α.
What is the relation between the s_k and the result of applying Gram-Schmidt to the sequence 1, x, x^2, x^3 and so on? Explain why that is the case.

12. Let $f_1, \cdots, f_t, f_{t+1}, \cdots, f_{t+u}$ be linear functionals on the finite dimensional real vector space V. Show that $Q(\mathbf{x}) = f_1(\mathbf{x})^2 + \cdots + f_t(\mathbf{x})^2 - f_{t+1}(\mathbf{x})^2 - \cdots - f_{t+u}(\mathbf{x})^2$ is a quadratic form on V. Suppose Q has rank $p + q$ and signature $p - q$. Show that $p \leq t$ and $q \leq u$.

13. Suppose that α is an orthogonal endomorphism on the finite-dimensional real inner product space V. Prove that V can be decomposed into a direct sum of mutually orthogonal α-invariant subspaces of dimension 1 or 2. Determine the possible matrices of α with respect to orthonormal bases in the cases where V has dimension 1 or dimension 2.

14. Let a_1, a_2, ... , a_n be real numbers such that $a_1 + \cdots + a_n = 0$ and $a_1^2 + \cdots + a_n^2 = 1$. What is the maximum value of $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1$?

15. Let $A = (a_{i,j})$ a square symetric matric of size $n \geq 1$. Given $1 \leq k \leq n$, we let $A_k = (a_{i,j})_{1 \leq i,j \leq k}$ and $d_k = \det A_k$.

    (a) Show that A is definite positive iff $d_k > 0$ for $1 \leq k \leq n$.

    (b) Let $0 < t < 1$. Show that $(t^{|i-j|})_{1 \leq i,j \leq n}$ is definite positive.

    (c) Show that the matrix $\left(\frac{1}{1+|i-j|}\right)_{1 \leq i,j \leq n}$ is definite positive.

16. Let E be a finite dimensional real euclidian space and p, q be two orthogonal projectors of E. Show that pq is diagonalizable with eigenvalues between 0 and 1.

17. Let $\mathcal{S}_n(\mathbb{R})$ denote the set of positive symmetric square matrices. An element $M \in \mathcal{S}_n(\mathbb{R})$ is called a state if $\mathrm{tr}M = 1$.

    (a) Characterize the states S, called pure states, such that $S^2 = S$. Show that $\forall X \in \mathbb{R}^n$, $\|SX\| \leq \|X\|$ with equality iff $SX = X$.

(b) A state is said to be extremal if it cannot be expressed as the barycenter with strictly positive coefficients of two other distinct states. Show that the set of extremal points are the pure states.

18. We equip $\mathbb{R}^n$ with the canonical scalar product $(\cdot|\cdot)$ and norm $\|\cdot\|$, and $\mathcal{M}_n(\mathbb{R})$ with the associated operator norm. We note S the unit sphere of $\mathbb{R}^n$. We let Γ_p be the set of vector subspaces of $\mathbb{R}^n$ which have dimension p. We let $\mathcal{S}_n(\mathbb{R})$ denote the set of symmetric square matrices.

    (a) Let $A \in \mathcal{S}_n(\mathbb{R})$ and $\lambda_1(A) \leq \ldots \leq \lambda_n(A)$ the ordered set of its eigenvalues. Show that $\lambda_p(A) = \min_{F \in \Gamma_p} \max_{X \in S \cap F} (AX|X)$.

    (b) Show that the map $A \mapsto \lambda_p(A)$ is 1-Lipschitz.

    (c) Let $B \in \mathcal{S}_n(\mathbb{R})$. Show that for $i + j \leq n + 1$, $\lambda_{i+j-1}(A + B) \geq \lambda_i(A) + \lambda_j(B)$.