Michaelmas Term 2022

Linear Algebra: Example Sheet 3 of 4

Exercices 12-13-14 are more difficult and optional.

1. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

   The second and third matrices commute; find a basis with respect to which they are both diagonal.

2. By considering the rank or minimal polynomial of a suitable matrix, find the eigenvalues of the $n \times n$ matrix A with each diagonal entry equal to λ and all other entries 1. Hence write down the determinant of A.

3. Let A be an $n \times n$ matrix all the entries of which are real. Show that the minimum polynomial of A, over the complex numbers, has real coefficients.

4. (i) Let V be a vector space, let $\pi_1, \pi_2, \dots, \pi_k$ be endomorphisms of V such that $\mathrm{id}_V = \pi_1 + \cdots + \pi_k$ and $\pi_i \pi_j = 0$ for any $i \neq j$. Show that $V = U_1 \oplus \cdots \oplus U_k$, where $U_j = \mathrm{Im}(\pi_j)$.

   (ii) Let α be an endomorphism of V satisfying the equation $\alpha^3 = \alpha$. By finding suitable endomorphisms of V depending on α, use (i) to prove that $V = V_0 \oplus V_1 \oplus V_{-1}$, where V_λ is the λ-eigenspace of α.

5. Let α be an endomorphism of a complex vector space. Show that if λ is an eigenvalue for α then λ^2 is an eigenvalue for α^2. Show further that every eigenvalue of α^2 arises in this way. Are the eigenspaces $\mathrm{Ker}(\alpha - \lambda\iota)$ and $\mathrm{Ker}(\alpha^2 - \lambda^2\iota)$ necessarily the same?

6. Without appealing directly to the uniqueness of Jordan Normal Form show that none of the following matrices are similar:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Is the matrix

$$\begin{pmatrix} -2 & -2 & -1 \\ 3 & 3 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

   similar to any of them? If so, which? Find a basis such that it is in Jordan Normal Form.

7. (a) Recall that the Jordan normal form of a 3×3 complex matrix can be deduced from its characteristic and minimal polynomials. Give an example to show that this is not so for 4×4 complex matrices.
   (b) Let A be a 5×5 complex matrix with $A^4 = A^2 \neq A$. What are the possible minimal polynomials of A? If A is not diagonalisable, what are the possible characteristic polynomials and JNFs of A?

8. Let V be a vector space of dimension n and α an endomorphism of V with $\alpha^n = 0$ but $\alpha^{n-1} \neq 0$. Without appealing to Jordan Normal Form, show that there is a vector y such that $(y, \alpha(y), \alpha^2(y), \dots, \alpha^{n-1}(y))$ is a basis for V. What is the matrix representation of α with respect to this basis? And the matrix representation of α^k, for an arbitrary positive integer k?

   Show that if β is an endomorphism of V which commutes with α, then $\beta = p(\alpha)$ for some polynomial p. *[Hint: consider $\beta(y)$.]* What is the form of the matrix for β with respect to the above basis?

9. (a) Let A be an invertible square matrix. Describe the eigenvalues and the characteristic and minimal polynomials of A^{-1} in terms of those of A.

   (b) Prove that the inverse of a Jordan block $J_m(\lambda)$ with $\lambda \neq 0$ has Jordan Normal Form a Jordan block $J_m(\lambda^{-1})$. Use this to find the Jordan Normal Form of A^{-1}, for an invertible square matrix A.

   (c) Prove that any square complex matrix is similar to its transpose.

10. Let C be an $n \times n$ matrix over $\mathbb{C}$, and write $C = A + iB$, where A and B are real $n \times n$ matrices. By considering $\det(A + \lambda B)$ as a function of λ, show that if C is invertible then there exists a real number λ such that $A + \lambda B$ is invertible. Deduce that if two $n \times n$ real matrices P and Q are similar when regarded as matrices over $\mathbb{C}$, then they are similar as matrices over $\mathbb{R}$.

11. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, with $a_i \in \mathbb{C}$, and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ a_n & a_0 & a_1 & \ldots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \ldots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}.$$

Show that the determinant of C is $\det C = \prod_{j=0}^{n} f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n+1))$.

12. Let E be a $\mathbb{C}$ vector space of dimension $n \geq 1$. We say that $u \in \mathcal{L}(E)$ is cyclic if there exists $x \in E$ such that $(x, u(x), \ldots, u^{n-1}(x))$ is a basis of E.

   (a) Let u nilpotent with $\dim\mathrm{Ker}(u) = 1$. Show that u is cyclic.

   (b) Assume a decomposition $E = V_1 \oplus \ldots \oplus V_r$ with $u(V_i) \subset V_i$ and $u_{|V_i}$ cyclic. Show that if the minimal polynomials of $u_{|V_i}$, $1 \leq i \leq r$, are prime to one another, then u is cyclic.

   (c) Let $A \in \mathcal{M}_n(\mathbb{C})$. Show that the following are equivalent.
   (a) The minimal polynomial of A is of degree n.
   (b) The eigenspaces of A are all one dimensional.
   (c) There exists $Y \in \mathbb{C}^n$ such that $(Y, \ldots, A^{n-1}Y)$ is a basis of $\mathbb{C}^n$.
   (d) The map $f_A : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ given by $(X, Y) \mapsto ({}^tXY, \ldots, {}^tXA^{n-1}Y)$ is surjective.
   (e) The set of vector subspaces of $\mathbb{C}^n$ which are stable by A is finite.

13. (a) Let E be a vector space over $\mathbb{C}$ of dimension $n \geq 1$. Let u be an endomorphism of E. Show (without using the Jordan decomposition) that there exists a unique couple (D, N) of endomorphisms of E such that $u = D + N$, D is diagonal, N is nilpotent and $DN = ND$.

   (b) Solve the equation $e^M = \mathrm{Id}$ in $\mathcal{M}_n(\mathbb{C})$.

   (c) Solve the equation $e^M = \mathrm{Id}$ in $\mathcal{M}_n(\mathbb{R})$

14. Let E be a vector space over $\mathbb{C}$ of dimension $n \geq 1$. Let u be an endomorphism of E. We define the subsets of $\mathcal{L}(E)$:

$$\mathbb{C}(u) = \{P(u), \quad P(X) \in \mathbb{C}[X]\} \subset \Gamma(u) = \{v \in \mathcal{L}(E), \quad vu = uv\}.$$

   (a) Show that $\dim\mathbb{C}(u) = \deg\mu_u$ where μ_u is the minimal polynomial of u.

   (b) Assume u is diagonalizable with distinct eigenvalues $(\lambda_1, \ldots, \lambda_r)$ of multiplicity $(n_1, \ldots, n_r)$. Show that

$$\dim\mathbb{C}(u) = r \quad \text{and} \quad \dim\Gamma(u) = \sum_{i=1}^{r} n_i^2.$$

   Conclude that

$$\dim\mathbb{C}(u) = n \Leftrightarrow \dim\Gamma(u) = n \Leftrightarrow r = n \Leftrightarrow \mathbb{C}(u) = \Gamma(u).$$

   (c) We no longer assume u diagonalizable. Show that $\dim\Gamma(u) \geq n$.

   (d) We say that u is cyclic if its minimal polynomial is of degree n. Show that u cyclic $\Leftrightarrow \Gamma(u) = \mathbb{C}(u)$. (*Hint: If u is cyclic, show that there exists* $X \in \mathbb{R}^n$ *such* $(X, u(X), \ldots, u^{n-1}(X))$ *is a basis of* E *and consider the linear map* $f : B \in \mathcal{C}(A) \mapsto BX \in \mathbb{R}^n$)