Michaelmas Term 2022

Linear Algebra: Example Sheet 2 of 4

Exercices 12-13-14-15 are more difficult and optional.

1. Write down the three types of elementary matrices and find their inverses. Use elementary matrices to find the inverse of
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}.$$

2. (Another proof of the row rank column rank equality.) Let A be an $m \times n$ matrix of (column) rank r. Show that r is the least integer for which A factorises as $A = BC$ with $B \in \mathrm{Mat}_{m,r}(\mathbb{F})$ and $C \in \mathrm{Mat}_{r,n}(\mathbb{F})$. Using the fact that $(BC)^T = C^T B^T$, deduce that the (column) rank of A^T equals r.

3. Let V be a 4-dimensional vector space over $\mathbb{R}$, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the basis of V^* dual to the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ for V. Determine, in terms of the ξ_i, the bases dual to each of the following:
(a) $\{\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_3\}$;
(b) $\{\mathbf{x}_1, 2\mathbf{x}_2, \frac{1}{2}\mathbf{x}_3, \mathbf{x}_4\}$;
(c) $\{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_3 + \mathbf{x}_4, \mathbf{x}_4\}$;
(d) $\{\mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 + \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_3 + \mathbf{x}_2 - \mathbf{x}_1\}$.

4. For $A \in \mathrm{Mat}_{n,m}(\mathbb{F})$ and $B \in \mathrm{Mat}_{m,n}(\mathbb{F})$, let $\tau_A(B)$ denote $\mathrm{tr}(AB)$. Show that, for each fixed A, $\tau_A\colon \mathrm{Mat}_{m,n}(\mathbb{F}) \to \mathbb{F}$ is linear. Show moreover that the mapping $A \mapsto \tau_A$ defines a linear isomorphism $\mathrm{Mat}_{n,m}(\mathbb{F}) \to \mathrm{Mat}_{m,n}(\mathbb{F})^*$.

5. (a) Suppose that $f \in \mathrm{Mat}_{n,n}(\mathbb{F})^*$ is such that $f(AB) = f(BA)$ for all $A, B \in \mathrm{Mat}_{n,n}(\mathbb{F})$ and $f(I) = n$. Show that f is the trace functional, i.e. $f(A) = \mathrm{tr}A$ for all $A \in \mathrm{Mat}_{n,n}(F)$.

   (b) Now let V be a non-zero finite dimensional real vector space. Show that there are no endomorphisms α, β of V with $\alpha\beta - \beta\alpha = \mathrm{id}_V$.

   (c) Let V be the space of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. Find endomorphisms α and β of V such that $\alpha\beta - \beta\alpha = \mathrm{id}_V$.

6. Suppose that $\psi\colon U \times V \to \mathbb{F}$ is a bilinear form of rank r on finite dimensional vector spaces U and V over $\mathbb{F}$. Show that there exist bases $e_1, \dots, e_m$ for U and $f_1, \dots, f_n$ for V such that
$$\psi\left(\sum_{i=1}^{m} x_i e_i, \sum_{j=1}^{n} y_j f_j\right) = \sum_{k=1}^{r} x_k y_k$$
for all $x_1, \dots, x_m, y_1, \dots, y_n \in \mathbb{F}$. What are the dimensions of the left and right kernels of ψ?

7. (a) Let $a_0, ..., a_n$ be distinct real numbers, and let
$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{pmatrix}.$$
Show that $\det(A) \neq 0$.

   (b) Let P_n be the space of real polynomials of degree at most n. For $x \in \mathbf{R}$ define $e_x \in P_n^*$ by $e_x(p) = p(x)$. By considering the standard basis $(1, t, \dots, t^n)$ for P_n, use (a) to show that $\{e_0, ..., e_n\}$ is linearly independent and hence forms a basis for P_n^*.

   (c) Identify the basis of P_n to which $(e_0, ..., e_n)$ is dual.

8. Let A, B be $n \times n$ matrices, where $n \geq 2$. Show that, if A and B are non-singular, then

$$(i)\ \mathrm{adj}\,(AB) = \mathrm{adj}\,(B)\mathrm{adj}\,(A)\,, \quad (ii)\ \det(\mathrm{adj}\,A) = (\det A)^{n-1}\,, \quad (iii)\ \mathrm{adj}\,(\mathrm{adj}\,A) = (\det A)^{n-2}A\,.$$

Show that the rank of the adjugate matrix is $\mathrm{r}(\mathrm{adj}\,A) = \begin{cases} n & \text{if } \mathrm{r}(A) = n \\ 1 & \text{if } \mathrm{r}(A) = n-1 \\ 0 & \text{if } \mathrm{r}(A) \leq n-2. \end{cases}$

Do (i)-(iii) hold if A is singular? *[Hint: for (i) consider $A + \lambda I$ for $\lambda \in \mathbb{F}$.]*

9. Show that the dual of the space P of real polynomials is isomorphic to the space $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers, via the mapping which sends a linear form $\xi : P \to \mathbb{R}$ to the sequence $(\xi(1), \xi(t), \xi(t^2), \ldots)$.

   In terms of this identification, describe the effect on a sequence $(a_0, a_1, a_2, \ldots)$ of the linear maps dual to each of the following linear maps $P \to P$:
   (a) The map D defined by $D(p)(t) = p'(t)$.
   (b) The map S defined by $S(p)(t) = p(t^2)$.
   (c) The composite DS.
   (d) The composite SD.

   Verify that $(DS)^* = S^*D^*$ and $(SD)^* = D^*S^*$.

10. Let V be a finite dimensional vector space. Suppose that $f_1, \ldots, f_n, g \in V^*$. Show that g is in the span of $f_1, \ldots, f_n$ if and only if $\bigcap_{i=1}^n \ker f_i \subset \ker g$. What if V is infinite dimensional?

11. Let $\alpha : V \to V$ be a linear map on a real finite dimensional vector space V with $\mathrm{tr}(\alpha) = 0$.
    (i) Show that, if $\alpha \neq 0$, there is a vector $\mathbf{v}$ with $\mathbf{v}, \alpha(\mathbf{v})$ linearly independent. Deduce that there is a basis for V relative to which α is represented by a matrix A with all of its diagonal entries equal to 0.
    (ii) Show that there are endomorphisms β, γ of V with $\alpha = \beta\gamma - \gamma\beta$.

12. Let A, H in $\mathcal{M}_n(\mathbb{R})$ with H of rank 1. Show that $\det(A+H)\det(A-H) \leq (\det A)^2$.

13. Let $A \in \mathcal{M}_n(\mathbb{R})$ such that $\forall 1 \leq i \leq n$, $|a_{i,i}| > \sum_{j \neq i} |a_{i,j}|$.

    (a) Show that A is invertible.

    (b) Assume moreover $\forall 1 \leq i \leq n$, $a_{i,i} \geq 0$. Is $\det A > 0$?

14. Let A, B in $\mathcal{M}_n(\mathbb{R})$ such that $\exists X \in \mathrm{Ker}A \backslash \{0\}$ with $BX \in \mathrm{Im}A$. Let A_i be the matrix obtained by replacing the i-th column of A by the i-th column of B. Show that $\sum_{i=1}^n \det A_i = 0$.

15. Let $M \in \mathcal{M}_n(\mathbb{R})$.

    (a) Let $\Delta_{i,j}$ be the minor of order i, j that is the determinant of the matrix obtained by removing the i-th line and the j-th column from M. More generally, let $\Delta_{I,J}$ be the determinant of the matrix obtained by removing all the lines $i \in I$ and columns $j \in J$ from M. Show that $\Delta_{\{1,n\},\{1,n\}}\det M = \Delta_{1,1}\Delta_{n,n} - \Delta_{1,n}\Delta_{n,1}$.
    (*Hint: consider the matrix M^* obtained by replacing in* $\mathrm{adj}(M)$ *the element (i,j) for $1 \leq i \leq n$, $2 \leq j \leq n-1$, by 0 if $i \neq j$ and 1 for $i = j$.*)

    (b) Let a real sequence $(a_k)_{k \geq 1}$ and let D_n be the determinant of the matrix $(a_{|i-j|})_{1 \leq i,j \leq n}$. Show that $D_{n-2}D_n \leq D_{n-1}^2$.