Michaelmas Term 2022

Linear Algebra: Example Sheet 1 of 4

Exercices 13-14-15-16 are more difficult and optional.

1. Suppose that the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form a basis for a real vector space V. Which of the following are also bases?

   (a) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n$;

   (b) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n + \mathbf{e}_1$;

   (c) $\mathbf{e}_1 - \mathbf{e}_n, \mathbf{e}_2 + \mathbf{e}_{n-1}, \ldots, \mathbf{e}_n + (-1)^n \mathbf{e}_1$.

2. Let T, U and W be subspaces of V.
   (i) Show that $T \cup U$ is a subspace of V only if either $T \leq U$ or $U \leq T$.
   (ii) Give explicit counter-examples to the following statements:

   (a) $T + (U \cap W) = (T + U) \cap (T + W)$; (b) $(T + U) \cap W = (T \cap W) + (U \cap W)$.

   (iii) Show that each of the equalities in (ii) can be replaced by a valid inclusion of one side in the other.

3. For each of the following pairs of vector spaces (V, W) over $\mathbb{R}$, either give an isomorphism $V \to W$ or show that no such isomorphism can exist. *[Here P denotes the space of polynomial functions $\mathbb{R} \to \mathbb{R}$, and $C[a, b]$ denotes the space of continuous functions defined on the closed interval $[a, b]$.]*
   (a) $V = \mathbb{R}^4$, $W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 0\}$.
   (b) $V = \mathbb{R}^5$, $W = \{p \in P : \deg\ p \leq 5\}$.
   (c) $V = C[0, 1]$, $W = C[-1, 1]$.
   (d) $V = C[0, 1]$, $W = \{f \in C[0, 1] : f(0) = 0, f \text{ continuously differentiable }\}$.
   (e) $V = \mathbb{R}^2$, $W = \{\text{real solutions of } \ddot{x}(t) + x(t) = 0\}$.
   (f) $V = \mathbb{R}^4$, $W = C[0, 1]$.
   (g) (Harder:) $V = P$, $W = \mathbb{R}^{\mathbb{N}}$.

4. (i) If α and β are linear maps from U to V show that $\alpha + \beta$ is linear. Give explicit counter-examples to the following statements:

   (a) $\mathrm{Im}(\alpha + \beta) = \mathrm{Im}(\alpha) + \mathrm{Im}(\beta)$; (b) $\mathrm{Ker}(\alpha + \beta) = \mathrm{Ker}(\alpha) \cap \mathrm{Ker}(\beta)$.

   Show that in general each of these equalities can be replaced by a valid inclusion of one side in the other.
   (ii) Let α be a linear map from V to V. Show that if $\alpha^2 = \alpha$ then $V = \mathrm{Ker}(\alpha) \oplus \mathrm{Im}(\alpha)$. Does your proof still work if V is infinite dimensional? Is the result still true?

5. Let

   $$U = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_3 + x_4 = 0,\ 2x_1 + 2x_2 + x_5 = 0\},\quad W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_5 = 0,\ x_2 = x_3 = x_4\}.$$

   Find bases for U and W containing a basis for $U \cap W$ as a subset. Give a basis for $U + W$ and show that

   $$U + W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + 2x_2 + x_5 = x_3 + x_4\}.$$

6. (i) Let $\alpha : V \to V$ be an endomorphism of a finite dimensional vector space V. Show that

   $$V \geq \mathrm{Im}(\alpha) \geq \mathrm{Im}(\alpha^2) \geq \ldots \quad \text{and} \quad \{0\} \leq \mathrm{Ker}(\alpha) \leq \mathrm{Ker}(\alpha^2) \leq \ldots.$$

   If $r_k = \mathrm{r}(\alpha^k)$, deduce that $r_k \geq r_{k+1}$ and that $r_k - r_{k+1} \geq r_{k+1} - r_{k+2}$. Conclude that if, for some $k \geq 0$, we have $r_k = r_{k+1}$, then $r_k = r_{k+\ell}$ for all $\ell \geq 0$.
   (ii) Suppose that $\dim(V) = 5$, $\alpha^3 = 0$, but $\alpha^2 \neq 0$. What possibilities are there for $\mathrm{r}(\alpha)$ and $\mathrm{r}(\alpha^2)$?

7. Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $\alpha : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find the matrix representing α relative to the basis $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for both the domain and the range.

   Write down bases for the domain and range with respect to which the matrix of α is the identity.

8. Let $U_1, \ldots, U_k$ be subspaces of a vector space V and let B_i be a basis for U_i. Show that the following statements are equivalent:

   (i) $U = \sum_i U_i$ is a direct sum, *i.e.* every element of U can be written uniquely as $\sum_i u_i$ with $u_i \in U_i$.

   (ii) $U_j \cap \sum_{i \neq j} U_i = \{0\}$ for all j.

   (iii) The B_i are pairwise disjoint and their union is a basis for $\sum_i U_i$.

   Give an example where $U_i \cap U_j = \{0\}$ for all $i \neq j$, yet $U_1 + \ldots + U_k$ is not a direct sum.

9. Show that any two subspaces of the same dimension in a finite dimensional real vector space have a common complementary subspace.

10. Let Y and Z be subspaces of the finite dimensional vector spaces V and W, respectively. Show that $R = \{\alpha \in \mathcal{L}(V, W) : \alpha(Y) \leq Z\}$ is a subspace of the space $\mathcal{L}(V, W)$ of all linear maps from V to W. What is the dimension of R?

11. Let Y and Z be subspaces of the finite dimensional vector spaces V and W respectively. Suppose that $\alpha\colon V \to W$ is a linear map such that $\alpha(Y) \subset Z$. Show that α induces linear maps $\alpha|_Y\colon Y \to Z$ via $\alpha|_Y(y) = \alpha(y)$ and $\overline{\alpha}\colon V/Y \to W/Z$ via $\overline{\alpha}(v + Y) = \alpha(v) + Z$.

    Consider a basis $(v_1, \ldots, v_n)$ for V containing a basis $(v_1, \ldots, v_k)$ for Y and a basis $(w_1, \ldots, w_m)$ for W containing a basis $(w_1, \ldots, w_l)$ for Z. Show that the matrix representing α with respect to $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ is a block matrix of the form $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Explain how to determine the matrices representing $\alpha|_Y$ with respect to the bases $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_l)$ and representing $\overline{\alpha}$ with respect to the bases $(v_{k+1} + Y, \ldots, v_n + Y)$ and $(w_{l+1} + Z, \ldots, w_m + Z)$ from this block matrix.

12. Let T, U, V, W be vector spaces over $\mathbb{F}$ and let $\alpha\colon T \to U$, $\beta\colon V \to W$ be fixed linear maps. Show that the mapping $\Phi\colon \mathcal{L}(U, V) \to \mathcal{L}(T, W)$ which sends θ to $\beta \circ \theta \circ \alpha$ is linear. If the spaces are finite-dimensional and α and β have rank r and s respectively, find the rank of Φ.

13. Let E be a complex vector space of dimension $n \geq 1$ and $u \in \mathcal{L}(E)$ be nilpotent of order p ($u^p = 0$ and $u^{p-1} \neq 0$). Show that there exists a decomposition $E = F_1 \oplus \ldots \oplus F_p$ with $u(F_1) = \{0\}$ and for $j \geq 2$, u induces an injection of F_j to F_{j-1}. Moreover, for $1 \leq j \leq p$, $\mathrm{Ker}(u^j) = F_1 \oplus \ldots \oplus F_j$.

14. (i) Let $\mathcal{G}$ be a subset of $\mathcal{M}_n(\mathbb{R})$ which is a group for the multiplication of matrices. Show that all elements of $\mathcal{G}$ have the same rank p and that there exists a basis $\mathcal{B}$ of $\mathbb{R}^n$ in which any element of $\mathcal{G}$ is represented by a matrix of the form $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ where A is $p \times p$.

    (ii) Let $M \in \mathcal{M}_n(\mathbb{R})$. Show that the following are equivalent:
    (a) M belongs to a group $\mathcal{G}$ as above.
    (b) M and M^2 have the same rank.
    (c) $\mathbb{R}^n = \mathrm{Im}M \oplus \mathrm{Ker}M$.

15. Let E be a complex vector space of dimension $n \geq 1$. Let f be an endomorphism of E. Show the existence and uniqueness of (F, G) vector subspaces of E such that:
    (a) $F \oplus G = E$;
    (b) F, G are stable by f;
    (c) $f_{|F}$ is nilpotent and $f_{|G}$ is invertible.
    (*Hint: consider the sequences* $F_k = \mathrm{Ker} f^k$ *and* $G_k = \mathrm{Im} f^k$.)

16. Let E be a complex vector space of dimension $n \geq 1$. Let f, g be two endomorphisms of E with $fg = gf$. Show that the following are equivalent:
(a) $\mathrm{Ker} f \cap \mathrm{Ker} g = \{0\}$;
(b) $\mathrm{Im} f + \mathrm{Im} g = E$;
(c) there exists a finite set S such that $\forall t \in \mathbb{C} \backslash S$, $f + tg$ is invertible.