Lecture 9

Properties of the dual map, double dual

- $\alpha \in L(V, W)$, $\alpha^* \in L(W^*, V^*)$

  $\mathcal{B}$ basis of V $\qquad$ $\mathcal{B}^*$ dual basis of V^*

  $\mathcal{C}$ basis of W $\qquad$ $\mathcal{C}^*$ dual basis of W^*

$$[\alpha^*]_{\mathcal{C}^*, \mathcal{B}^*} = [\alpha]_{\mathcal{B}, \mathcal{C}}^T$$

- $\mathcal{E} = (e_1, \dots, e_n)$

  $\mathcal{F} = (f_1, \dots, f_n)$

$P = [Id]_{\mathcal{F}, \mathcal{E}}$ (change of basis matrix from $\mathcal{F}$ to $\mathcal{E}$)

$\mathcal{E}^* = (\varepsilon_1, \dots, \varepsilon_n)$, $\mathcal{F}^* = (\eta_1, \dots, \eta_n)$

the corresponding dual basis.

Lemma Change of basis matrix from $\mathcal{F}^*$ to $\mathcal{E}^*$ is: $\left(\underline{P}^{-1}\right)^T$

proof $[Id]_{\mathcal{F}^*,\mathcal{E}^*} = [Id]_{\mathcal{E},\mathcal{F}}^T$

$$= \left([Id]_{\mathcal{F},\mathcal{E}}^{-1}\right)^T = \left(\underline{P}^{-1}\right)^T \qquad \square$$

Properties of the dual map

Lemma Let V, W be vector spaces over F. Let $\alpha \in L(V,W)$. Let $\alpha^* \in L(W^*, V^*)$ be the dual map. Then:

(i) $N(\alpha^*) = (\operatorname{Im} \alpha)^\circ$
(so α^* injective $\Leftrightarrow$ α surjective)

(ii) $\operatorname{Im} \alpha^* \leq (N(\alpha))^\circ$
with equality if V and W are finite dimensional (hence α^* surjective $\Leftrightarrow$ α injective in this case)

$\Rightarrow$ VERY IMPORTANT PROPERTY

$\rightsquigarrow$ lots of concrete applications: it is often simpler to understand α^* than α.
(duality approach)

proof (i) Let $\varepsilon \in W^*$. Then:

$$\varepsilon \in N(\alpha^*) \Leftrightarrow \alpha^*(\varepsilon) = 0$$

$$\Leftrightarrow \quad \alpha^*(\varepsilon) = \varepsilon \circ \alpha = 0$$

$$\Leftrightarrow \quad \forall x \in V, \quad \varepsilon(\alpha(x)) = 0$$

$$\Leftrightarrow \quad \varepsilon \in (\operatorname{Im} \alpha)^\circ .$$

(ii) Let us first show that $\boxed{\operatorname{Im}(\alpha^*) \leq (N(\alpha))^\circ}$.

Indeed, let $\varepsilon \in \operatorname{Im}(\alpha^*)$

$$\Rightarrow \quad \varepsilon = \alpha^*(\varphi), \quad \varphi \in W^*$$

$$\Rightarrow \quad \forall u \in N(\alpha), \quad \varepsilon(u) = \alpha^*(\varphi)(u)$$

$$= \varphi \circ \alpha(u) = \varphi(\underbrace{\alpha(u)}_{=0}) = 0$$

$$\Rightarrow \quad \varepsilon \in (N(\alpha))^\circ .$$

In finite dimension, we can compare dimensions:

$$\dim \operatorname{Im}(\alpha^*) = r(\alpha^*) = r(\alpha)$$

$$r(\alpha^*) = r\left([\alpha^*]_{\mathcal{C}^*, \mathcal{B}^*}\right) = r\left([\alpha]_{\mathcal{B}, \mathcal{C}}^T\right) = r(\alpha)$$

$$\left(r(A) = r(A^T) \right)$$

$$\Rightarrow \dim \operatorname{Im} \alpha^* = r(\alpha^*) = r(\alpha)$$

$$= \dim V - \dim N(\alpha)$$

$$= \dim (N(\alpha))^\circ$$

$$\Rightarrow \quad \operatorname{Im} \alpha^* \leq (N(\alpha))^\circ$$

$$\dim \operatorname{Im} \alpha^* = \dim (N(\alpha)^\circ) \quad \Bigg| \Rightarrow$$

$$\operatorname{Im} \alpha^* = (N(\alpha))^\circ \qquad \square$$

Double dual $\quad V$ vector space over F

$V^* = L(V, F)$ dual of V

We define the bidual:

$$V^{**} = L(V^*, F) = (V^*)^*$$

→ very important space : in general, there is no obvious relation between V and V^*.

<u>However</u>, there is a $\boxed{\text{canonical embedding}}$ of V into V^{**}.

Indeed, pick $v \in V$, let :

$$\hat{v} : V^* \longrightarrow F$$
$$\varepsilon \longmapsto \varepsilon(v) = \hat{v}(\varepsilon)$$

<u>Claim</u> $\hat{v} \in V^{**}$

. $\varepsilon \in V^* \Rightarrow \varepsilon(v) \in F$

. $\lambda_1, \lambda_2 \in F$, $\varepsilon_1, \varepsilon_2 \in V^*$,

$$\hat{v}(\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2) \underset{\text{def}}{=} (\lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2)(v)$$

$$= \lambda_1 \varepsilon_1(v) + \lambda_2 \varepsilon_2(v)$$
$$= \lambda_1 \hat{v}(\varepsilon_1) + \lambda_2 \hat{v}(\varepsilon_2)$$, linearity.

Theorem If V is a finite dimensional subspace over F, then:

$$\hat{\ } : V \longrightarrow V^{**}$$
$$v \longmapsto \hat{v}$$

is an isomorphism.

proof . $\hat{v} \in V^{**}$ we just showed this.

. $\hat{\ }$ linear $v_1, v_2 \in V$, $\lambda_1, \lambda_2 \in F$, $\varepsilon \in V^*$,

$$\widehat{\lambda_1 v_1 + \lambda_2 v_2}(\varepsilon)$$

$$= \varepsilon(\lambda_1 v_1 + \lambda_2 v_2)$$

$$= \lambda_1 \varepsilon(v_1) + \lambda_2 \varepsilon(v_2)$$

$$= \lambda_1 \hat{v}_1(\varepsilon) + \lambda_2 \hat{v}_2(\varepsilon)$$

$$\Rightarrow \widehat{\lambda_1 v_1 + \lambda_2 v_2} = \lambda_1 \hat{v}_1 + \lambda_2 \hat{v}_2$$

. $\widehat{\ }$ injective. Indeed, let $e \in V \setminus \{0\}$.

I extend e to a basis of V:

$(e, e_2, \ldots, e_n)$ basis of V.

Let $(\varepsilon, \varepsilon_2, \ldots, \varepsilon_n)$ be the dual basis (of V^*), then:

$$\hat{e}(\varepsilon) = \varepsilon(e) = 1$$

$$\Rightarrow \hat{e} \neq \{0\}$$

$$\Rightarrow N(\wedge) = \{0\}, \ \wedge \text{ injective.}$$

. $\hat{}$ isomorphism

$$\dim V = \dim V^* = \dim (V^*)^* = \dim V^{**}$$

$$\hat{} : V \to V^{**} \quad \text{injective}$$

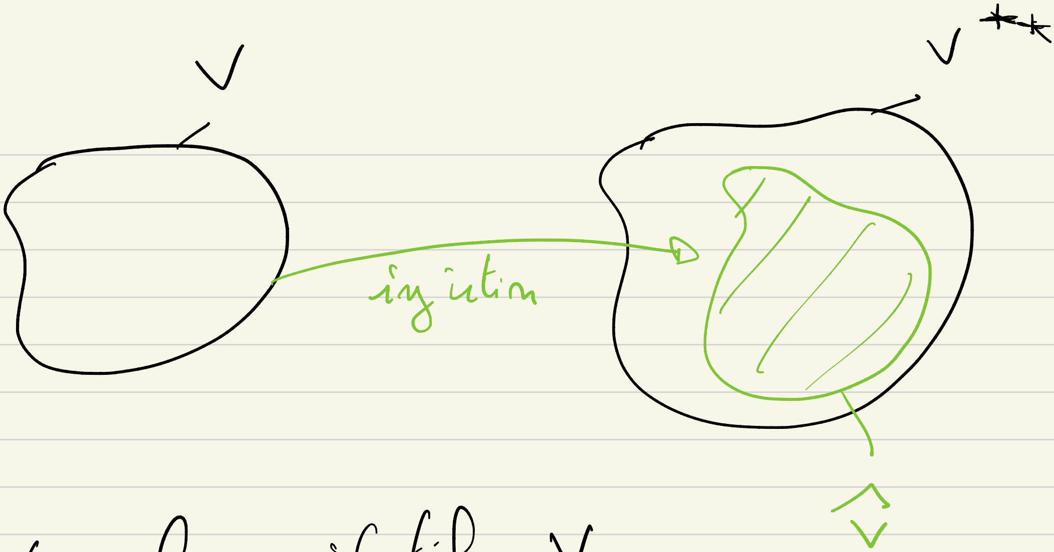
$$\dim V = \dim V^{**}$$

$\Longrightarrow$ $\hat{}$ isomorphism. $\square$

Cultural remark $\Longrightarrow$ breakthrough mathematics from the 1950's

$\to$ functional analysis.

. $\hat{}$ injective : remains true for a large class of infinite dimensional spaces



We can always identify V as a subset of V**.

Then There are many infinite dimensional spaces for which the injection is again an isomorphism: reflexive spaces.

$\Rightarrow$ many applications in analysis.

(linear analysis / analysis of functions)

V finite dimensional $\wedge$ isomorphism so I can identify V and V**.

$\boxed{\text{Lemma}}$ Let V be a finite dimensional vector space over F, and $U \leq V$. Then:

$$\widehat{U} = U^{\circ\circ},$$

so after identification of V and V^{**},

$$\boxed{U^{\circ\circ} = U}.$$

proof. Let us show that: $U \leq U^{\circ\circ}$.

Indeed, let $u \in U$:

$\Rightarrow \forall \varepsilon \in U^{\circ}, \quad \varepsilon(u) = 0 \quad (\text{def } U^{\circ})$

$\Rightarrow \forall \varepsilon \in U^{\circ}, \quad \varepsilon(u) = \hat{u}(\varepsilon) = 0$

$\Rightarrow \quad \hat{u} \in U^{\circ\circ}$

$\Rightarrow \quad \widehat{U} \subset U^{\circ\circ}$

. I compute dimensions:

$\dim U^{\circ\circ} = \dim V - \dim U^{\circ} = \dim U$

$\Rightarrow$ $\wedge$ isomorphism, $\dim \hat{U} = \dim U$

$\Rightarrow$ $\dim U = \dim U^{\circ\circ}$

$\Rightarrow$ $\hat{U} = U^{\circ\circ}$

$\|$ $\leftarrow$ identify V and V^{**}

U

Rk $T \leq V^*$, we can define:

$$T^{\circ} = \{v \in V \,/\, \theta(v) = 0, \ \forall \theta \in T\}$$

Lemma Let V be finite dimensional vector space over F. Let $U_1, U_2 \leq V$. Then:

(i) $(U_1 + U_2)^{\circ} = U_1^{\circ} \cap U_2^{\circ}$

(ii) $(U_1 \cap U_2)^{\circ} = U_1^{\circ} + U_2^{\circ}$

proof (i) let $\vartheta \in V^*$

$\vartheta \in (U_1+U_2)^\circ \iff \forall (u_1, u_2) \in U_1 \times U_2$

$\vartheta(u_1+u_2) = 0$

$\iff \vartheta(u) = 0 \quad \forall u \in U_1 \cup U_2$ (linearity of ϑ)

$\iff \vartheta \in U_1^\circ \cap U_2^\circ$

We have proved $(U_1+U_2)^\circ = U_1^\circ \cap U_2^\circ$

(ii) Take $^\circ$ of (i) and use: $U^{\circ\circ} = U$

$\rightsquigarrow$ bilinear maps