Lecture 8 — Dual spaces and dual maps

Def V F vector space

$V^* =$ dual of V

$= L(V, F) = \{ \alpha : V \to F \text{ linear} \}$

Notation $\alpha : V \to F$ linear

$\equiv$ linear form

Ex (i) $Tr : M_{n,n}(F) \to F$

$$A = (a_{ij}) \mapsto \sum_{i=1}^{n} a_{ii}$$

$Tr \in M_{n,n}^*$

(ii) $f : [0,1] \to \mathbb{R}$

$x \mapsto f(x)$

$$T_f : \mathcal{C}^\infty([0,1], \mathbb{R}) \longrightarrow \mathbb{R}$$
$$\varphi \longmapsto \int_0^1 f(x)\varphi(x)\,dx$$

T_f linear form on $\mathcal{C}^\infty([0,1], \mathbb{R})$

$\rightsquigarrow$ you can reconstruct f from T_f.

Lemma (Def) Let V be a vector space over F, with a finite basis

$\mathcal{B} = \{e_1, \dots, e_n\}$

Then there exists a basis for V^* given by:

$\mathcal{B}^* = \{\varepsilon_1, \dots, \varepsilon_n\}$ where:

$$\boxed{\varepsilon_j\left(\sum_{i=1}^{n} a_i e_i\right) = a_j}, \quad 1 \leq j \leq n$$

$\mathcal{B}^*$ = dual basis of $\mathcal{B}$.

Remark Kronecker symbol :

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon_j\left(\sum_{i=1}^{n} a_i e_i\right) = a_j$$

$$\iff \boxed{\varepsilon_j(e_i) = \delta_{ij}}$$

proof . $(\varepsilon_1, \dots, \varepsilon_n)$ free.

$$\sum_{j=1}^{n} \lambda_j \varepsilon_j = 0 \Rightarrow \sum_{j=1}^{n} \lambda_j \overbrace{\varepsilon_j(e_i)}^{\delta_{ij}} = 0 \quad \forall i \in \{1, \dots, n\}$$

$\Rightarrow \lambda_i = 0, \ \forall 1 \leq i \leq n$.

. Span $\alpha \in V^*$,

$$\alpha(x) = \alpha\Big(\overbrace{\sum_{j=1}^{n} \lambda_j e_j}^{x}\Big) = \sum_{j=1}^{n} \lambda_j \alpha(e_j)$$

On the other hand, let : $\boxed{\sum_{j=1}^{n} \alpha(e_j)\, \varepsilon_j} \in V^*$,

$$\sum_{j=1}^{n} \alpha(e_j)\varepsilon_j(x) = \sum_{j=1}^{n} \alpha(e_j)\varepsilon_j\Big(\sum_{k=1}^{n} \lambda_k e_k\Big)$$

$$= \sum_{j=1}^{n} \alpha(e_j) \sum_{k=1}^{n} \underbrace{\varepsilon_j(e_k)}_{\delta_{jk}} \lambda_k$$

$$= \sum_{j=1}^{n} \alpha(e_j)\lambda_j = \alpha(x)$$

$$\Rightarrow \boxed{\alpha = \sum_{j=1}^{n} \alpha(e_j)\,\varepsilon_j}$$

$\square$

$\boxed{\textbf{Cor}\quad V \text{ finite dimensional: } \dim V^* = \dim V}$

Remark It is sometimes convenient to think of V^* as the space of row vectors of length n over F.

$e_1, \dots, e_n$ basis of V, $x = \sum x_i e_i \in V$

$\varepsilon_1, \dots, \varepsilon_n$ basis of V^*, $\alpha = \sum \alpha_i \varepsilon_i \in V^*$

$$\alpha(x) = \sum_{i=1}^{n} \alpha_i \varepsilon_i \left(\sum_{j=1}^{n} x_j e_j \right)$$

$$= \sum_{i,j} \alpha_i x_j \underbrace{\varepsilon_i(e_j)}_{\delta_{ij}} = \sum_{i=1}^{n} \alpha_i x_i$$

$$= \begin{pmatrix} \alpha_1 & \dots & \alpha_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} \alpha_i x_i$$

$\rightarrow$ scalar product structure.

Def If $U \subseteq V$, the annihilator of U is:

subset

$$U^0 = \{ \alpha \in V^* \,/\, \forall u \in U,\ \alpha(u) = 0 \}$$

lemma (i) $U^0 \leq V^*$ (vector subspace)

(ii) If $U \leq V$ and $\dim V < +\infty$,

$$\dim V = \dim U + \dim U^0$$

proof (i) . $0 \in U^0$

. If $\alpha, \alpha' \in U^0$ then:

$$\forall u \in U,\ (\alpha + \alpha')(u) = \underset{\substack{\shortparallel \\ 0}}{\alpha(u)} + \underset{\substack{\shortparallel \\ 0}}{\alpha'(u)} = 0$$

and: $\forall \lambda \in F$, $(\lambda\alpha)(u) = \lambda\alpha(u) = 0$

$\Rightarrow U^\circ \leq V^*$.

(iii) Let $U \leq V$, $\dim V = n$. Let $(e_1, \dots, e_k)$ basis of U, complete it to a basis $\underbrace{(e_1, \dots, e_k, e_{k+1}, \dots, e_n)}_{B}$ of V.

Let $\underbrace{(\varepsilon_1, \dots, \varepsilon_n)}_{B^*}$ be the dual basis of B

Claim $U^\circ = \langle \varepsilon_{k+1}, \dots, \varepsilon_n \rangle$

- If $i > k$, $\varepsilon_i(e_k) = \delta_{ik} = 0$

$\Longrightarrow \varepsilon_i \in U^\circ$ $\quad (U = \langle e_1, \dots, e_k \rangle)$

$\Rightarrow \langle \varepsilon_{k+1}, \dots, \varepsilon_n \rangle \subset U^\circ$

. let $\alpha \in U^0$ then: $\alpha = \sum_{i=1}^{n} \alpha_i \varepsilon_i$

(B^* basis of V^*)

Now for $i \leq k$:

$$\alpha \in U^0 \Rightarrow \alpha(e_i) = 0$$

$$\Rightarrow \alpha(e_i) = \sum_{j=1}^{n} \alpha_j \underbrace{\varepsilon_j(e_i)}_{\delta_{ij}} = \alpha_i$$

$$\Rightarrow \alpha_i = 0, \quad \forall 1 \leq i \leq k$$

$$\boxed{\varepsilon_j(e_i) = \delta_{ij}}$$

$$\Rightarrow \quad \alpha = \sum_{i=k+1}^{n} \alpha_i \varepsilon_i$$

$$\Rightarrow \quad \alpha \in \langle \varepsilon_{k+1}, \dots, \varepsilon_n \rangle.$$

$$\Rightarrow \quad U^0 \subset \langle \varepsilon_{k+1}, \dots, \varepsilon_n \rangle$$

Def / Lemma V, W vector spaces over F. Let $\alpha \in L(V, W)$. Then the map:

$$\alpha^* : W^* \longrightarrow V^* \quad \text{is}$$

$$\varepsilon \longmapsto \varepsilon \circ \alpha$$

an element of $L(W^*, V^*)$. It is called the dual map of α.

proof . $\varepsilon \circ \alpha : V \longrightarrow F$ linear by linearity of ε and α

$\Longrightarrow \varepsilon \circ \alpha \in V^*$

. α^* linear: pick $\theta_1, \theta_2 \in W^*$, then:

$$\alpha^*(\theta_1 + \theta_2) \underset{\substack{\uparrow \\ \text{def}}}{=} (\theta_1 + \theta_2) \circ \alpha = \theta_1 \circ \alpha + \theta_2 \circ \alpha$$

$$= \alpha^*(\theta_1) + \alpha^*(\theta_2)$$

and similarly: $\forall \lambda \in F$

$$\alpha^*(\lambda\theta) = \lambda\alpha^*(\theta)$$

$\Rightarrow$ α^* linear, $\alpha^* \in L(W^*, V^*)$.

Prop Let V, W finite dimensional space over F with basis $\mathcal{B}, \mathcal{C}$. Let $\mathcal{B}^*, \mathcal{C}^*$ be the dual basis of V^*, W^*, then:

$$[\alpha^*]_{\mathcal{C}^*, \mathcal{B}^*} = [\alpha]_{\mathcal{B}, \mathcal{C}}^{T}$$

Dual Map $\Longleftrightarrow$ Adjoint of α

proof $\mathcal{B} = (b_1, \dots, b_n)$, $\mathcal{C} = (c_1, \dots, c_m)$

$\mathcal{B}^* = (\beta_1, \dots, \beta_n)$, $\mathcal{C}^* = (\gamma_1, \dots, \gamma_m)$

Say: $[\alpha]_{\mathcal{B},\mathcal{C}} = A = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}$ $\Leftarrow$.

Let's compute: $\quad (\alpha^* : W^* \to V^*)$

$$\alpha^*(\gamma_r)(b_s)$$

$$= \gamma_r \circ \alpha(b_s) = \underbrace{\gamma_r}_{\in W^*}\big(\overbrace{\alpha(b_s)}^{W}\big)$$

$$= \gamma_r\Big(\underbrace{\sum_t \underbrace{a_{ts} c_t}}_{s\text{-th column vector}}\Big) = \sum_t a_{ts} \overbrace{\gamma_r(c_t)}^{\delta_{rt}}$$

$$= a_{rs}$$

$$\left(A = [\alpha]_{\mathcal{B},\mathcal{C}} = \begin{array}{c} \left(\alpha(b_1) \,\middle|\, \dots \,\middle|\, \alpha(b_n) \right) \end{array} \begin{array}{c} c_1 \\ \vdots \\ c_m \end{array} \right)$$

Say : $[\alpha^*]_{\mathcal{C}^*, \mathcal{B}^*} = \begin{array}{c} \\ \left(\alpha^*(\gamma_1) \middle| \cdots \middle| \alpha^*(\gamma_m) \right) \end{array} \begin{array}{c} \beta_1 \\ \vdots \\ \beta_n \end{array}$

$$\underset{\text{def}}{\equiv} (m_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le m}}$$

Then $\alpha^*(\gamma_r) = \sum_{i=1}^{n} \underbrace{m_{ir}\, \beta_i}_{r\text{th column vector}}$

$$\Longrightarrow \alpha^*(\gamma_r)(b_s) = \sum_{i=1}^{n} m_{ir} \underbrace{\beta_i(b_s)}_{\delta_{is}} = m_{sr}$$

Conclusion $\alpha^*(\gamma_r)(b_s) = a_{rs} = m_{sr}$

$$\Longleftrightarrow [\alpha^*]_{\mathcal{C}^*, \mathcal{B}^*} = [\alpha]_{\mathcal{B}, \mathcal{C}}^{T} \qquad \square.$$