Lecture 7 — Elementary operations and elementary matrices

Special case of the change of basis formula

- $\alpha : V \longrightarrow W$ linear

  (B, B') basis of V

  $(\mathcal{C}, \mathcal{C}')$ basis of W

  $$[\alpha]_{B, \mathcal{C}} \rightsquigarrow [\alpha]_{B', \mathcal{C}'}$$

- $V = W$ $\quad \alpha : V \longrightarrow V$ linear

  (= endomorphism)

- $B = \mathcal{C} \qquad B' = \mathcal{C}'$
- P = change of matrix from B' to B

then $\quad [\alpha]_{B', B'} = P^{-1} [\alpha]_{B, B} P$

Def A, A' $n \times n$ (square) matrices

We say that A and A' are similar (≡ conjugate) iff:

$$\begin{cases} A' = P^{-1} A P \\ P \equiv n \times n \text{ square invertible} \end{cases}$$

→ Central concept when dealing with diagonalization of endomorphisms

⤳ SPECTRAL THEORY.

Elementary operations and elementary matrices

Def Elementary column operation on an $m \times n$ matrix A :

(i) swap columns i and j $(i \neq j)$

(ii) replace column i by $\lambda \times$ column i
$(\lambda \neq 0, \ \lambda \in F)$

(iii) add $\lambda \times$ column i to column j $(i \neq j)$

- Elementary row operations : analogous way
- These elementary operations are invertible.
- These operations can be realized through

The action of elementary matrices :

(i) i, j, $i \neq j$

$$T_{ij} = \begin{array}{cl}
\begin{pmatrix}
1 &&&&&&&& \\
& \ddots &&&&&&& \\
&& 1 &&&&&& \\
&&& 0 & \cdots & 1 &&& \\
&&& \vdots & 1 & \vdots &&& \\
&&& \vdots & \ddots & \vdots &&& \\
&&& \vdots & 1 & \vdots &&& \\
&&& 1 & \cdots & 0 &&& \\
&&&&&& 1 && \\
&&&&&&& \ddots & \\
&&&&&&&& 1
\end{pmatrix}
&
\begin{array}{l}
\\ \\ \\ \leftarrow i \\ \\ \\ \\ \leftarrow j \\ \\ \\ \\
\end{array}
\end{array}$$

(ii)

$$M_{i,\lambda} = \begin{array}{cl}
\begin{pmatrix}
1 &&&&& \\
& \ddots &&& 0 & \\
&& 1 &&& \\
&&& \lambda && \\
& 0 &&& 1 & \\
&&&&& \ddots \\
&&&&&& 1
\end{pmatrix}
&
\begin{array}{l}
\\ \\ \\ \leftarrow i \\ \\ \\ \\
\end{array}
\end{array}$$

(iii) $C_{i,j,\lambda} = I_d + \lambda E_{ij}$

$$E_{ij} = \begin{array}{cc} \begin{array}{ccc} & j & \end{array} & \\ \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} & \begin{array}{c} \\ i \\ \\ \end{array} \end{array}$$

Links between elementary operations and matrices:

an elementary column (row) operation can be performed by multiplying A by the corresponding elementary matrix from the right (left).

Exercise

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$

Constructive proof that any $m \times n$ matrix is equivalent to:

$$\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) \quad \text{for some } r.$$

- Start with A. If all entries are zero, done.
- Pick $a_{ij} = \lambda \neq 0$:
  - . swap rows i and 1
  - . swap columns j and 1

$\Rightarrow$ get λ in position $(1,1)$

- Multiply column 1 by $1/\lambda$ $\quad (\lambda \neq 0)$

$\Rightarrow$ get 1 in position $(1,1)$

- Now clear out row 1 and column 1 using elementary operations of type (iii):

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \tilde{A} & \\ 0 & & & \end{pmatrix}$$

- Continue with $\tilde{A}$ $(m-1)\times(n-1)$.
- End of the process:

$$\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) = Q^{-1} A P.$$

$$= \underbrace{\overbrace{E'_\ell \cdots E'_1}^{Q^{-1}}}_{\text{row operations}} A \underbrace{\overbrace{E_1 \cdots E_c}^{P}}_{\text{column operations}}$$

Variations

① Gauss' pivot algorithm

If you use <u>only</u> row operations you can reach the so called "row echelon form" of the matrix:

"pivot"

$$\begin{pmatrix} 0 & \cdots & 0 & 1 & * & \cdots & \cdots & * \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & * \cdots & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

- Assume that $a_{i1} \neq 0$ for some i
- Swap rows i and 1
- Divide first row by $\lambda = a_{i1}$ (refers to initial numbering) to get 1 in $(1,1)$.
- Use 1 to clean the rest of the 1st column.

- move to 2d column
- iterate

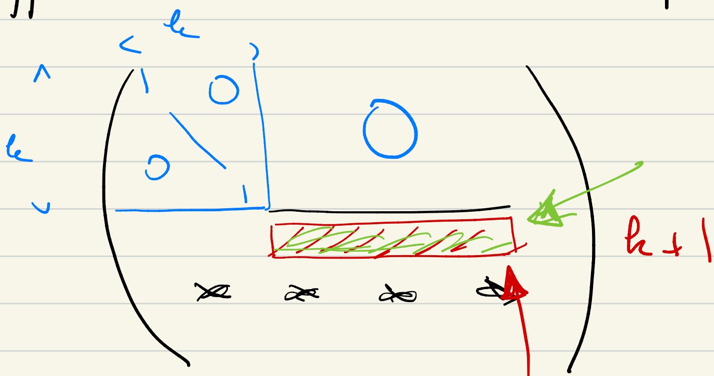
$\Rightarrow$ This procedure is exactly what you do when solving a linear system of equations (Gauss' pivot algorithm)

② Representation of square invertible matrices.

Lemma If A is a $n \times n$ (square) invertible matrix, then we can obtain I_n using row elementary operations only (resp. column operations only)

proof We do the proof for column operations.
We argue by induction on the number of rows.

. Suppose that we could reach the form:



. I want to obtain the same structure with k+1 rows.

. **Claim** $\exists j > k \;/\; a_{k+1,j} = \lambda > 0$.

Indeed, otherwise we can show that:

$$\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix} \leftarrow k+1$$

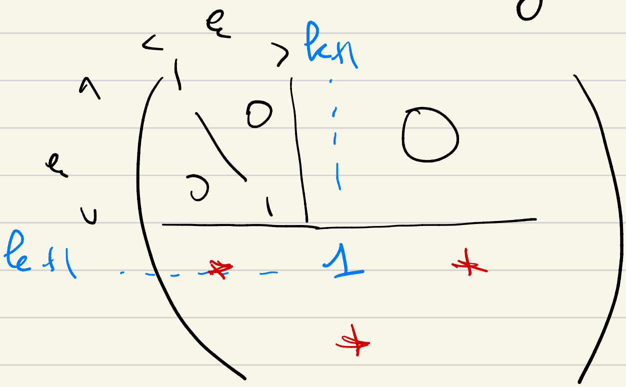
NOT in the span of the column vectors of A.

(this is an exercise)

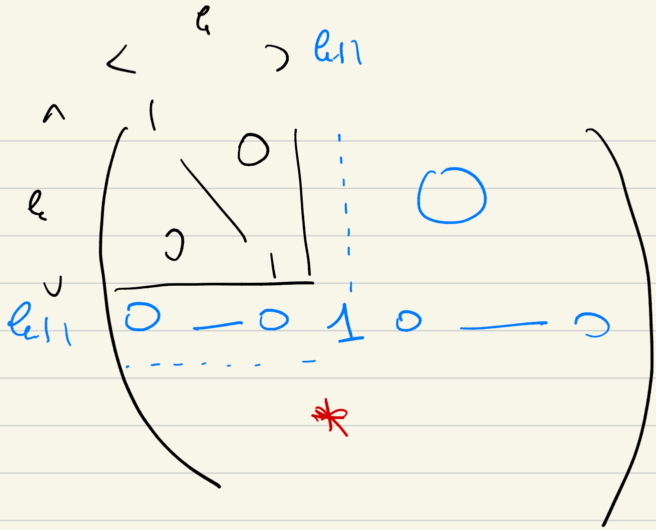
This contradicts our assumption that A is invertible.

. swap column $k+1$ and j

. divide column $k+1$ by $\lambda = a_{k+1,j} \neq 0$



. use 1 to clear the rest of the $(k+1)$ row using type (iii) elementary operations.



$\rightsquigarrow$ structure for the induction hypothesis at the order $k+1$.

Outcome $\quad AE_1 \dots E_c = I_n$

$\Rightarrow \quad A^{-1} = E_1 \dots E_c$

$\Rightarrow$ algorithm for computing A^{-1}

($\Leftrightarrow$ algorithm for solving the linear system of n unknowns with n equations:

$$AX = F \Leftrightarrow X = A^{-1}F \;)$$

$\rightsquigarrow$ Constructive way to compute the inverse of A.

Prop Any invertible square matrix is a product of elementary matrices.