Lecture 6 Change of basis and equivalent matrices

- $\alpha : V \to W$

  $\mathcal{B}$ basis of V, $\mathcal{C}$ basis of W

  $$\boxed{[\alpha(v)]_{\mathcal{C}} = [\alpha]_{\mathcal{B},\mathcal{C}} \cdot [v]_{\mathcal{B}}}$$

  $$[\alpha]_{\mathcal{B},\mathcal{C}} = \begin{array}{c} \left(\alpha(v_1) \middle| \ldots \middle| \alpha(v_n) \right) \end{array}\begin{array}{c} w_1 \\ \vdots \\ w_m \end{array}$$

- $U \xrightarrow{\beta} V \xrightarrow{\alpha} W$

  $\mathcal{A}, \mathcal{B}, \mathcal{C}$ basis U, V, W

$$\Rightarrow \boxed{[\alpha \circ \beta]_{\mathcal{A},\mathcal{C}} = [\alpha]_{\mathcal{B},\mathcal{C}} [\beta]_{\mathcal{A},\mathcal{B}}}$$

Change basis

$$V \xrightarrow{\alpha} W$$

$\mathcal{B} = \{v_1, \dots, v_n\}$ $\qquad$ $\mathcal{C} = \{w_1, \dots, w_m\}$

$\mathcal{B}' = \{v_1', \dots, v_n'\}$ $\qquad$ $\mathcal{C}' = \{w_1', \dots, w_m'\}$

Aim $\quad [\alpha]_{\mathcal{B},\mathcal{C}} \quad , \quad [\alpha]_{\mathcal{B}',\mathcal{C}'}$

Def The "change of basis" matrix from $\mathcal{B}'$ to $\mathcal{B}$ is $P = (p_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}}$

given by :

$$P = \left([v_1']_{\mathcal{B}} \,\middle|\, [v_2']_{\mathcal{B}} \,\middle|\, \dots \,\middle|\, [v_n']_{\mathcal{B}} \right)$$

$$\overset{\text{def}}{\equiv} [\mathrm{Id}]_{\mathcal{B}', \mathcal{B}}$$

Lemma $\boxed{[v]_{\mathcal{B}} = P[v]_{\mathcal{B}'}}$ $(*)$

proof. $[\alpha(v)]_{\mathcal{C}} = [\alpha]_{\mathcal{B},\mathcal{C}}[v]_{\mathcal{B}}$

. $P = [\mathrm{Id}]_{\mathcal{B}',\mathcal{B}}$

$$\Rightarrow \quad [\mathrm{Id}(v)]_{\mathcal{B}} = [\mathrm{Id}]_{\mathcal{B}',\mathcal{B}}[v]_{\mathcal{B}'}$$

$$(\mathcal{B}=\mathcal{C},\ \mathcal{B}'=\mathcal{B})$$

$$\Rightarrow \quad [v]_{\mathcal{B}} = P[v]_{\mathcal{B}'}.$$

$\square$.

Remark P is an $n\times n$ invertible matrix, and: P^{-1} = change of basis matrix from to $\mathcal{B}'$.

Indeed: $[\alpha \circ \beta]_{A,\mathcal{C}} = [\alpha]_{B,\mathcal{C}} \, [\beta]_{A,B}$

$$[\mathrm{Id}]_{B,B'} \, [\mathrm{Id}]_{B',B} = [\mathrm{Id}]_{B',B'} = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$[\mathrm{Id}]_{B',B} \, [\mathrm{Id}]_{B,B'} = [\mathrm{Id}]_{B,B} = I_n$$

$$\Rightarrow [\mathrm{Id}]_{B,B'} \, \underline{P} = \underline{P} \, [\mathrm{Id}]_{B,B'} = I_n . \qquad \blacksquare$$

⚠ $[v]_B$, $\underline{P} = \left([v_1']_B, \ldots, [v_n']_B \right)$

$$[v]_{B'} = \underline{P}^{-1} [v]_B .$$

$\Rightarrow$ need to invert $\underline{P}$!.

. We changed $\mathcal{B}$ to $\mathcal{B}'$ in V.

. We can also change basis $\mathcal{C}$ to $\mathcal{C}'$ in W

$$\left(\alpha : V \longrightarrow W \right)$$

V	W	P $n \times n$ Q $m \times m$ $\Downarrow$
$\mathcal{B}, \mathcal{B}'$	$\mathcal{C}, \mathcal{C}'$	
$P = [\mathrm{Id}]_{\mathcal{B}', \mathcal{B}}$	$Q = [\mathrm{Id}]_{\mathcal{C}', \mathcal{C}}$	Invertible matrices

$$\alpha : V \longrightarrow W$$

$$[\alpha]_{\mathcal{B}, \mathcal{C}} \overset{?}{\longleftrightarrow} [\alpha]_{\mathcal{B}', \mathcal{C}'}$$

?..

$\boxed{\text{Prop}}$

$$A = [\alpha]_{\mathcal{B}, \mathcal{C}}, \quad A' = [\alpha]_{\mathcal{B}', \mathcal{C}'}$$

$$P = [\mathrm{Id}]_{\mathcal{B}', \mathcal{B}} \quad Q = [\mathrm{Id}]_{\mathcal{C}', \mathcal{C}}$$

$$\boxed{\Longrightarrow \quad A' = Q^{-1} A \underline{P}.}$$

<u>proof</u>

$$[\alpha(v)]_{\mathcal{C}} = [\alpha]_{\mathcal{B},\mathcal{C}} [v]_{\mathcal{B}}$$

$$[\alpha \circ \beta]_{\mathcal{A},\mathcal{C}} = [\alpha]_{\mathcal{B},\mathcal{C}} [\beta]_{\mathcal{A},\mathcal{B}}$$

$$[v]_{\mathcal{B}} = \underline{P} [v]_{\mathcal{B}'}$$

- $$[\alpha(v)]_{\mathcal{C}} = Q [\alpha(v)]_{\mathcal{C}'}$$

$$= Q [\alpha]_{\mathcal{B}',\mathcal{C}'} [v]_{\mathcal{B}'} = Q A' [v]_{\mathcal{B}'}$$

- $$[\alpha(v)]_{\mathcal{C}} = [\alpha]_{\mathcal{B},\mathcal{C}} [v]_{\mathcal{B}}$$

$$= A \underline{P} [v]_{\mathcal{B}'}$$

$$\Longrightarrow \forall v \in V, \quad Q A' [v]_{\mathcal{B}'} = A \underline{P} [v]_{\mathcal{B}'}$$

$$\Rightarrow \quad QA' = AP$$

$$\Rightarrow \quad \boxed{A' = Q^{-1} A P}. \qquad \square$$

Def (Equivalent matrices)

Two matrices $A, A' \in \mathcal{M}_{m,n}(F)$ are equivalent if:

$$\begin{cases} A' = Q^{-1} A P \\ Q \in \mathcal{M}_{m \times m} \text{ invertible} \\ P \in \mathcal{M}_{n \times n} \text{ invertible} \end{cases}$$

Remark This defines an equivalence relation on $\mathcal{M}_{m,n}(F)$.

- $A = I_m^{-1} A I_n$

$$\left(I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right)$$

- $A' = Q^{-1} A P \Rightarrow A = (Q^{-1})^{-1} A^{-1} P^{-1}$
- $A' = Q^{-1} A P$

$$\left. \begin{array}{l} A'' = (Q')^{-1} A' P' \end{array} \right| \Rightarrow$$

$$A'' = (Q Q')^{-1} A (P P')$$

Prop Let V, W vector spaces over F.
$\dim_F V = n$, $\dim_F W = m$.
Let $\alpha : V \to W$ linear map. Then there exist $\mathcal{B}$ basis of V, $\mathcal{C}$ basis of W s.t.:

$$[\alpha]_{\mathcal{B}, \mathcal{C}} = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right), \quad I_r = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

proof Chose $\mathcal{B}$ and $\mathcal{C}$ wisely.

- Fix $r \in \mathbb{N}$ such that:

$$= \dim \ker \alpha = n - r.$$

- $N(\alpha) = \ker(\alpha) = \{x \in V \mid \alpha(x) = 0\}$.
- Fix a basis of $N(\alpha)$: $v_{r+1}, \dots, v_n$.
- Extend it to a basis of $V \equiv \mathcal{B}$

$$\mathcal{B} = (v_1, \dots, v_r, \underbrace{v_{r+1}, \dots, v_n}_{\ker \alpha})$$

Claim $(\alpha(v_1), \dots, \alpha(v_r))$ basis of $\operatorname{Im} \alpha$

- Span $v = \sum_{i=1}^{n} \lambda_i v_i$

$$\Rightarrow \alpha(v) = \sum_{i=1}^{n} \lambda_i \alpha(v_i)$$

$$= \sum_{i=1}^{r} \lambda_i \alpha(v_i)$$

Let $y \in \operatorname{Im}(\alpha)$, then: $\exists v \in V /\ \alpha(v) = y$

$\Rightarrow$
$$y = \alpha(v) = \sum_{i=1}^{r} \lambda_i \alpha(v_i)$$
$$\in \operatorname{Span}\{\alpha(v_1), \dots, \alpha(v_r)\}.$$

. <u>Free</u> $\displaystyle\sum_{i=1}^{r} \lambda_i \alpha(v_i) = 0$

$$\Rightarrow \quad \alpha\left(\sum_{i=1}^{r} \lambda_i v_i\right) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{r} \lambda_i v_i \in \ker \alpha = \operatorname{Span}\{v_{r+1}, \dots, v_n\}$$

$$\Rightarrow \quad \sum_{i=1}^{r} \lambda_i v_i = \sum_{i=r+1}^{n} \varphi_i v_i$$

$$\Rightarrow \quad \sum_{i=1}^{r} \lambda_i v_i - \sum_{i=r+1}^{n} \varphi_i v_i = 0$$

$\Rightarrow \quad \lambda_i = \varphi_i = 0 \quad \Rightarrow$ free.

We have proved that:

$\alpha(v_1), \dots, \alpha(v_r)$ basis of $\operatorname{Im}\alpha$

$v_{r+1}, \dots, v_n$ basis of $\operatorname{Ker}\alpha$

$\mathcal{B} = (v_1, \dots, v_r, v_{r+1}, \dots, v_n)$ basis of V

$\mathcal{C} = (\underbrace{\alpha(v_1), \dots, \alpha(v_r)}_{\operatorname{Im}\alpha}, w_{r+1}, \dots, w_m)$ basis of W

$$[\alpha]_{\mathcal{B}\mathcal{C}} = \left(\overbrace{\alpha(v_1) \,\Big|\, \dots \,\Big|\, \alpha(v_r)}^{r} \,\Big|\, \overbrace{\alpha(v_{r+1}) \,\Big|\, \dots \,\Big|\, \alpha(v_n)}\right) \begin{array}{l} \alpha(v_1) \\ \vdots \\ \alpha(v_r) \\ w_{r+1} \\ \vdots \\ w_n \end{array}$$

$$= \begin{array}{c} \\ r \\ \\ \end{array} \begin{array}{cc} \overbrace{\quad}^{r} & \\ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} & \end{array}$$

$n-r$

Rk This provides another proof of the rank nullity theorem:

$$r(\alpha) + N(\alpha) = n$$

Cor Any $m \times n$ matrix is equivalent to:

$$\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) \quad \text{where } r = r(\alpha)$$

Def $A \in M_{m,n}(F)$

. The column rank of A, $r(A)$, is the dimension of the subspace of F^m spanned by the column vectors of A

$$A = (c_1 | \ldots | c_n) \qquad c_i \in F^m$$

$r(A) = \dim_F \operatorname{Span}\{c_1, \dots, c_n\}$

- Similarly, the row rank of A is the column rank of A^T.

Rk If α is a linear map represented by A with respect to some basis, then:

$$r(A) = r(\alpha)$$

(column rank = rank)

Prop Two matrices are equivalent iff $r(A) = r(A')$

proof $\Rightarrow$) If A, A' equivalent, then they correspond to the same endomorphism α

expressed in two different basis:

$$\begin{cases} r(A) = r(u) \\ r(A') = r(\alpha) \end{cases}$$

does not depend on any basis

$\Rightarrow \quad r(A) = r(A')$

$\Leftarrow$) $r(A) = r(A') = r$, then A and A' are both equivalent to:

$$\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

$\Rightarrow$ A and A' are equivalent

$$\boxed{\textbf{Thm} \quad r(A) = r(A^T) \quad (\text{column rank} = \text{row rank})}$$

proof $r = r(A)$

$$Q^{-1} A P = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right)_{m \times n}$$

Take the transpose :

$$(Q^{-1} A P)^T = P^T A^T (Q^{-1})^T$$

$$= P^T A^T (Q^T)^{-1}$$

$$\left((AB)^T = B^T A^T \right)$$

$$\Rightarrow P^T A^T (Q^T)^{-1} = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right)_{m \times n}^T$$

$$= \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right)_{n \times m}$$

$$\Rightarrow \boxed{r(A^T) = r(A)}$$