Lecture 5 Linear maps from V to W and matrices

The space of linear maps from V to W

V, W vector spaces over F.

$$\boxed{L(V,W) = \{ \alpha : V \to W \text{ linear} \}}$$

Prop $L(V,W)$ is a vector space over F under the operations:

$$(\alpha_1 + \alpha_2)(v) \overset{\text{def}}{=} \alpha_1(v) + \alpha_2(v)$$

$$(\lambda \alpha)(v) \overset{\text{def}}{=} \lambda \alpha(v) \qquad \lambda \in F$$

Moreover, if V and W are finite dimensional over F, then so is $L(V,W)$ and:

$$\dim_F L(V,W) = (\dim_F V)(\dim_F W)$$

Proof. $L(V,W)$ vector space $\Rightarrow$ exercise

. About dimension ~ proved by the end of the lecture. □

Matrices and linear maps

Def An $m \times n$ matrix over F is an array with m rows and n columns, with entries in F:

$$A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} = \begin{array}{cc} & \overset{\longleftarrow \quad n \quad \longrightarrow}{\quad j \quad} \\ i & \begin{pmatrix} & \vdots & \\ \cdots & a_{ij} & \end{pmatrix} \end{array} \begin{matrix} \uparrow \\ m \\ \downarrow \end{matrix}$$

$a_{ij} \in F$.

row column

<u>Notation</u> $M_{m,n}(F) \equiv \{$ set of $m \times n$ matrices over $F \}$

Prop $M_{m,n}(F)$ is an F vector space under operations:

$$(a_{ij}) + (b_{ij}) \overset{\text{def}}{=} (a_{ij} + b_{ij})$$

$$\lambda (a_{ij}) = (\lambda a_{ij}) \quad , \quad \lambda \in F .$$

<u>proof</u> Exercise.

Prop $\dim_F M_{m,n}(F) = m \times n$

proof Basis: $1 \le i \le m$, $1 \le j \le n$, define "elementary matrix":

$$E_{ij} = \begin{array}{cc} & \quad j \\ i & \underbrace{\begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix}}_{n} \end{array} \quad m$$

Then $(E_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ basis of $M_{m,n}(F)$:

- Spans $M_{m,n}(F)$ obvious: $M = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij}$

- free family ⇝ obvious as well.

Representation of linear maps by matrices

- V, W vector spaces over F.
  $\alpha : V \to W$ linear.
- Basis: $\mathcal{B} = (v_1, \dots, v_n)$ of V
  $\mathcal{C} = (w_1, \dots, w_m)$ of W
- If $v \in V$, $v = \sum_{j=1}^{n} \lambda_j v_j = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \in F^n$

coordinates of v in the basis $\mathcal{B}$

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = [v]_{\mathcal{B}}$$

. Similarly for $\underline{w} \in W$, we note:

$$[w]_{\mathcal{C}} = \text{vector of coordinates of } w \text{ in the basis } \mathcal{C}.$$

Def (Matrix of α in $\underline{B}$, $\mathcal{C}$ basis)

$$[\alpha]_{\underline{B},\mathcal{C}} = \text{matrix of } \alpha \text{ wrt } \underline{B}, \mathcal{C}$$

$$\underset{\text{def}}{=} \left(\overbrace{[\alpha(v_1)]_{\mathcal{C}}, [\alpha(v_2)]_{\mathcal{C}}, \dots, [\alpha(v_n)]_{\mathcal{C}}}^{n}\right)\Big\} m$$

$$\in M_{m\times n}(F)$$

Observation $\quad [\alpha]_{\underline{B},\mathcal{C}} = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}$

Then by definition: $1 \le j \le n$

$$\alpha(v_j) = \sum_{i=1}^{m} a_{ij}\, w_i$$

$$\begin{array}{cc} j & \\ \left(\begin{array}{ccc} & \vdots & \\ & [\alpha(v_j)]_{\mathcal{C}} & \\ & \vdots & \end{array}\right) & \begin{array}{c} w_1 \\ \vdots \\ w_m \end{array} \end{array}$$

$$\boxed{\textbf{Lemma} \quad \forall\, v \in V, \quad [\alpha(v)]_{\mathcal{C}} = [\alpha]_{\mathcal{B}\,\mathcal{C}} \cdot [v]_{\mathcal{B}}.}$$

Remainder:

$$A = \overbrace{\begin{pmatrix} & & \\ & a_{ij} & \\ & & \end{pmatrix}}^{n} \Big\} m \qquad \begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{pmatrix} \equiv v \qquad \begin{pmatrix} \vdots \\ \cdot \\ \vdots \end{pmatrix} \leftarrow i$$

$$(Av)_i \overset{\text{def}}{=} \boxed{\sum_{j=1}^{n} a_{ij} \lambda_j}$$

proof $v \in V$, $v = \sum_{j=1}^{n} \lambda_j v_j$

$$\alpha(v) = \alpha\left(\sum_{j=1}^{n} \lambda_j v_j\right)$$

$$= \sum_{j=1}^{n} \lambda_j \alpha(v_j) = \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{m} a_{ij} w_i$$

$$= \sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{n} a_{ij} \lambda_j \right)} w_i$$

$$\left((a_{ij}) = [\alpha]_{\mathcal{B}, \mathcal{C}} \right)$$

Lemma $U \xrightarrow{\beta} V \xrightarrow{\alpha} W$ linear

$U \xrightarrow{\alpha \circ \beta} W$

$\mathcal{A}$ basis of U

$\mathcal{B}$ ——— of V

$\mathcal{C}$ ——— of W

product of matrices.

$$\Rightarrow [\alpha \circ \beta]_{\mathcal{A}, \mathcal{C}} = \underbrace{[\alpha]_{\mathcal{B}, \mathcal{C}}}_{A} \cdot \underbrace{[\beta]_{\mathcal{A}, \mathcal{B}}}_{B}$$

<u>proof</u> $u_\ell \in \mathcal{A}$

$$(\alpha \circ \beta)(v_\ell) = \alpha(\beta(v_\ell))$$

$$= \alpha\Big(\sum_j b_{j\ell} v_j\Big) \leftarrow \text{in } \mathcal{B}$$

$$= \sum_j b_{j\ell}\, \alpha(v_j) = \sum_j b_{j\ell} \sum_i a_{ij} w_i$$

in $\mathcal{C}$

$$= \sum_i \underbrace{\Big(\sum_j a_{ij} b_{j\ell}\Big)}_{(i,j) \text{ entry of } A.B} w_i$$

by definition of the product of matrices.

Prop If V and W are vector spaces over F,

$$\dim_F V = n$$

$$\dim_F W = m$$

$$\Longrightarrow \quad L(V,W) \simeq M_{m,n}(F) \quad (\text{isomorphic})$$

$$\left(\Longrightarrow \dim_F L(V,W) = m \times n \right)$$

proof Fix $\mathcal{B}, \mathcal{C}$ basis of resp V, W.

Claim
$$\theta : L(V,W) \longrightarrow M_{m,n}(F)$$
$$\alpha \longmapsto [\alpha]_{\mathcal{B},\mathcal{C}}$$

is an isomorphism.

. θ linear :

$$[\lambda_1 \alpha_1 + \lambda_2 \alpha_2]_{\mathcal{B},\mathcal{C}} = \lambda_1 [\alpha_1]_{\mathcal{B},\mathcal{C}} + \lambda_2 [\alpha_2]_{\mathcal{B},\mathcal{C}}$$

$\leadsto$ exercise -

. <u>θ surjective</u> Indeed, pick $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$

Consider the map:

$$\alpha : \quad v_j \longmapsto \sum_{i=1}^{m} a_{ij} w_i \quad , \quad 1 \le j \le n .$$

α is a map defined on $(v_1, \dots, v_m)$ = basis of V

$\Longrightarrow$ "extend by linearity"

$\Longrightarrow$ α linear map, which by definition:

$$[\alpha]_{\mathcal{B}, \mathcal{C}} = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} = A .$$

. <u>θ injective</u> $[\alpha]_{\mathcal{B}, \mathcal{C}} = 0 \Rightarrow \alpha \equiv 0 .$

Remark

$\mathcal{B}$ basis of V
$\mathcal{C}$ ——— W

$\varepsilon_{\mathcal{B}} : V \longrightarrow F^n$ $\qquad$ $\varepsilon_{\mathcal{C}} : W \longrightarrow F^m$

$v \longmapsto [v]_{\mathcal{B}}$ $\qquad$ $w \longmapsto [w]_{\mathcal{C}}$

Then the following diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "\alpha"] \arrow[d, "\varepsilon_{\mathcal{B}}"'] & W \arrow[d, "\varepsilon_{\mathcal{C}}"] \\
F^n \arrow[r, "{[\alpha]_{\mathcal{B},\mathcal{C}}}"] & F^m
\end{tikzcd}$$

Example

$\alpha : V \longrightarrow W$

$U \leq V$, $\boxed{\alpha(U) \leq Z}$ $(\leq W)$

. $\mathcal{B}$ basis for V, $(\underbrace{v_1, \dots, v_k}_{\text{basis of } U \,=\, \mathcal{B}'}, \underbrace{v_{k+1}, \dots, v_n}_{\mathcal{B}''})$

. Base de W $\mathcal{C}$:

$$\Big(\underbrace{w_1, \dots, w_\ell}_{\text{base de } Z \,=\, \mathcal{C}'}, \underbrace{w_{\ell+1}, \dots, w_m}_{\mathcal{C}''}\Big)$$

$$[\alpha]_{\underline{\mathcal{B}}\,\mathcal{C}} = \begin{array}{c} \\ \Big(\underbrace{\alpha(v_1) \,|\, \dots \,|\, \alpha(v_\ell)}_{\in \underline{Y}} \,|\, \alpha(v_{\ell+1}) \dots \,|\, \alpha(v_n) \Big) \end{array} \begin{array}{l} Z \\ \boxed{\begin{array}{l} w_1 \\ \vdots \\ w_\ell \end{array}} \\ w_{\ell+1} \\ \vdots \\ w_m \end{array}$$

$$= \begin{array}{c|c} & \overset{\ell}{\longleftrightarrow} \\ \ell \updownarrow & \left(\begin{array}{c|c} A & B \\ \hline O & C \end{array}\right) \end{array}$$

. $A = [\alpha_{|Y}]_{\underline{\mathcal{B}'}\,\mathcal{C}'}$ ← exercice.

("bloc A")

Exercise

α induces:

$$\bar{\alpha} : V/\underline{Y} \longrightarrow W/Z$$
$$v + \underline{Y} \longmapsto \alpha(v) + Z$$

. well defined: $v_1 + \underline{Y} = v_2 + \underline{Y}$

$\Rightarrow v_1 - v_2 \in \underline{Y}$

$\Rightarrow \alpha(v_1 - v_2) \in Z$

$\Rightarrow \alpha(v_1) + Z = \alpha(v_2) + Z$

. $\bar{\alpha}$ linear obvious (α linear)

. $[\bar{\alpha}]_{B'', \mathcal{C}''} = C$