Lecture 4 Linear maps, isomorphisms and the rank-nullity Theorem

Def (linear map) V, W are $\mathbb{F}$-vector spaces.
A map $\alpha : V \to W$ is <u>linear</u> iff:

$$\forall (\lambda_1, \lambda_2) \in \mathbb{F}^2, \ \forall (v_1, v_2) \in V \times V,$$

$$\alpha(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2)$$

Example

(i)

$$\begin{array}{cc} & \begin{pmatrix} x \end{pmatrix}\ \updownarrow m \end{array}$$

$$M = \begin{array}{cc} & \langle\, m \,\rangle \\ \updownarrow n & \begin{pmatrix} \quad & \quad \\ & \end{pmatrix} \end{array}$$

$$\alpha : \mathbb{R}^m \to \mathbb{R}^n \qquad \text{linear.}$$

$$x \mapsto Mx$$

(ii) $\alpha : \mathcal{C}([0;1]) \longrightarrow \mathcal{C}([0;1])$

$f \longmapsto \alpha(f)(x) = \int_0^x f(t)\,dt$

linear map.

(iii) Fix $x \in [a;b]$, $\mathcal{C}([a;b]) \longrightarrow \mathbb{R}$,

$f \longmapsto f(x)$

linear map.

Remark U, V, W F vector spaces

(i) $Id_V : V \longrightarrow V$, $x \mapsto x$ linear map

(ii)
$$\begin{tikzcd} U \arrow[r, "\beta"'] \arrow[rr, bend right, "\text{linear}"'] & V \arrow[r, "\alpha"'] & W \end{tikzcd}$$

linear linear

Linearity is stable by composition

Lemma V, W F vector spaces.
B basis for V.
If $\alpha_0 : B \to V$ is ANY map, then
there is a unique linear map $\alpha : V \to W$
extending α_0 (ie $\forall v \in B, \alpha_0(v) = \alpha(v)$)

proof $v \in V$, $\quad v = \sum_{i=1}^{n} \lambda_i v_i$

$B = (v_1, \dots, v_n)$

Necessarily, by linearity:

$$\alpha(v) = \alpha\left(\sum_{i=1}^{n} \lambda_i v_i\right) = \sum_{i=1}^{n} \lambda_i \alpha(v_i)$$

Remark (i) True for ∞ dimensional spaces as well
Often, to define a linear map, we define its
value on a basis and "extend by linearity".

(iii) $\alpha_1, \alpha_2 : V \to W$ linear. If they agree on a basis $\mathcal{B}$ of V, they are equal.

Def (Isomorphism) V, W vector spaces over F.
A map $\alpha : V \to W$ is called an <u>isomorphism</u> iff:

(i) α linear

(ii) α bijection

If such an α exists, we note: $V \simeq W$
(V is isomorphic to W)

<u>Rk</u> $\alpha : V \to W$ isomorphism
$\Longrightarrow \alpha^{-1} : W \to V$ is <u>linear</u>.

Indeed, $\left|\begin{array}{l}\alpha : V \to W \\ \alpha^{-1} : W \to V\end{array}\right.$

$(w_1, w_2) \in W \times W$, $\left(w_1 = \alpha(v_1), w_2 = \alpha(v_2) \right)$

$$
\begin{aligned}
& \alpha^{-1}(w_1 + w_2) \\
&= \alpha^{-1}\left(\alpha(v_1) + \alpha(v_2) \right) \\
&= \alpha^{-1}\left(\alpha(v_1 + v_2) \right) = v_1 + v_2 \\
&= \alpha^{-1}(w_1) + \alpha^{-1}(w_2).
\end{aligned}
$$

Similarly, $\forall (\lambda, w) \in F \times W$,

$$\alpha^{-1}(\lambda w) = \lambda \alpha^{-1}(w).$$

Lemma $\simeq$ is an equivalence relation on the class of all vector spaces over F.

(i) $i_V : V \to V$ isomorphism

(ii) $\alpha : V \to W$ isomorphism $\Rightarrow \alpha^{-1} : W \to V$ isomorphism

(iii) If $U \xrightarrow[\beta]{} V \xrightarrow[\alpha]{} W$, $\underline{\beta}$, $\underline{\alpha}$ isomorphic

$\Longrightarrow U \xrightarrow[\alpha \circ \beta]{} W$, $\alpha\circ\beta$ isomorphic

Proof Exercise.

Thm If V is a vector space over F of dimension n then:

$$V \cong F^n$$

$$\left(F^n = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i \in F, \ 1 \leq i \leq n \right\} \right)$$

proof Let $B = (v_1, \dots, v_n)$ be a basis of V.
Then:

$$\alpha : V \longrightarrow F^n$$

$$v = \sum_{i=1}^{n} \lambda_i v_i \longmapsto \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

is an isomorphism.

$\rightarrow$ exercise.

Rk Choosing a basis of V is like choosing an isomorphism from V to F^n.

Thm Let V, W F vector spaces with finite dimension. Then $V \simeq W$ iff they have the same dimension.

proof $\Leftarrow$) $\dim V = \dim W = n$

$\Rightarrow \begin{cases} V \simeq F^n \\ W \simeq F^n \end{cases} \Rightarrow V \simeq W.$

$\Rightarrow$) $\alpha : V \to W$ isomorphism, $\mathcal{B}$ is a basis for V, then:

claim $\alpha(\mathcal{B})$ basis for W.

. $\alpha(\mathcal{B})$ spans V follows from surjectivity of α

. $\alpha(\mathcal{B})$ free family ——— injectivity of α

$\longrightarrow$ exercise.

Def (Kernel and image of a linear map)

Let V, W vector spaces over F.

Let $\alpha : V \to W$ linear map.

We define :

- $N(\alpha) = \operatorname{Ker} \alpha = \{ v \in V \,/\, \alpha(v) = 0 \}$

  ($\equiv$ Kernel of α)

- $\operatorname{Im}(\alpha) = \{ w \in W \,/\, \exists v \in V ,\ w = \alpha(v) \}$

  ($\equiv$ Image of α)

Lemma Ker α and Im α are subspaces of respectively V and W.

proof $(\lambda_1, \lambda_2) \in F^2$, $(v_1, v_2) \in \operatorname{Ker}\alpha \times \operatorname{Ker}\alpha$,

$$\alpha(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \underset{\substack{\parallel \\ 0}}{\alpha(v_1)} + \lambda_2 \underset{\substack{\parallel \\ 0}}{\alpha(v_2)}$$

$$= 0$$

$\Rightarrow \lambda_1 v_1 + \lambda_2 v_2 \in \ker \alpha$.

. $(\lambda_1, \lambda_2) \in F^2$, $(w_1, w_2) \in (\mathrm{Im}\,\alpha)^2$,

$w_1 = \alpha(v_1)$, $w_2 = \alpha(v_2)$

$$\Rightarrow \lambda_1 w_1 + \lambda_2 w_2 = \lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2)$$
$$= \alpha(\lambda_1 v_1 + \lambda_2 v_2)$$
$$\in \mathrm{Im}\,\alpha$$

D.

Ex $\alpha : \mathcal{C}^\infty(\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R})$

$$f(t) \longmapsto \alpha(f)(t) = f''(t) + f(t)$$

. $\ker \alpha = \{ f \in \mathcal{C}^\infty(\mathbb{R}),\ f'' + f = 0 \}$

$$= \mathrm{Span}_{\mathbb{R}} \langle e^{t}, e^{-t} \rangle$$

. $\mathrm{Im}\,\alpha = ?$.

Th Let V, W $\mathbb{F}$ vector spaces.
Let $\alpha : V \to W$ linear map. Then:

$$V/\ker\alpha \xrightarrow{\bar{\alpha}} \operatorname{Im}\alpha$$

$$\bar{\alpha}(v + \ker\alpha) \mapsto \alpha(v)$$

is an isomorphism.

proof • $\bar{\alpha}$ well defined: $v + \ker\alpha = v' + \ker\alpha$
$\Rightarrow v - v' \in \ker\alpha \Rightarrow$
$\alpha(v - v') = 0 \Rightarrow \alpha(v) = \alpha(v')$.
$\Rightarrow \bar{\alpha}(v) = \bar{\alpha}(v')$

• $\bar{\alpha}$ linear follows immediately from α linear.

• $\bar{\alpha}$ bijection:

– injectivity: $\bar{\alpha}(v + \ker\alpha) = 0$

$\Rightarrow \quad \alpha(v) = 0 \Rightarrow v \in \operatorname{Ker}\alpha \Rightarrow$

$$v + \operatorname{Ker}\alpha = 0 + \operatorname{Ker}\alpha .$$

- surjectivity follows from the definition of $\operatorname{Im}\alpha$:

$w \in \operatorname{Im}\alpha, \quad \exists v \in V \,/\, \quad w = \alpha(v) = \bar{\alpha}(v)$

Def (Rank and nullity)

- $r(\alpha) = rk(\alpha) = \dim \operatorname{Im}(\alpha)$ (rank)
- $n(\alpha) = \dim \operatorname{Ker}(\alpha)$ (nullity)

Th (Rank - nullity Theorem)

- Let U, V be vector spaces over $\mathbb{F}$,

$$\dim_{\mathbb{F}} U < +\infty .$$

Let $\alpha : U \to V$ be a linear map, then:

$$\dim U = r(\alpha) + n(\alpha)$$

proof We have proved that:

$$U/_{\ker \alpha} \simeq \mathrm{Im}(\alpha)$$

$$\Rightarrow \quad \dim (U/\ker \alpha) = \dim \mathrm{Im}\, \alpha$$

$$\dim (U/\ker\alpha) = \dim U - \dim \ker \alpha$$

$$\Rightarrow \quad \dim U = r(\alpha) + n(\alpha) \qquad \square.$$

Lemma (Characterization of isomorphism)

V, W vector spaces over F of EQUAL FINITE DIMENSION. Let $\alpha : V \to W$ linear map, then TFAE:

$$\boxed{\begin{array}{ll} \text{(i)} & \alpha \text{ injective} \\ \text{(ii)} & \alpha \text{ surjective} \\ \text{(iii)} & \alpha \text{ isomorphism} \end{array}}$$

<u>proof</u> $\to$ exercise. Follows directly from the rank-nullity theorem.

<u>Ex</u>

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,/\, x+y+z = 0 \right\}$$

$$\dim V = ? \, .$$

$$\alpha : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto x+y+z$$

$\ker \alpha = V$

$\operatorname{Im} \alpha = \mathbb{R}$

$$\Rightarrow 3 = n(\alpha) + 1 \quad , \quad n(\alpha) = 2 .$$

(V is a plane)