Lecture 3 | Bases, dimension, direct sums

Thm (Steinitz Exchange lemma) V a finite dimensional vector space over F. Let $(v_1, \ldots, v_m)$ linearly independent ($\equiv$ free family), and $(w_1, \ldots, w_n)$ spans V. Then:

(i) $m \leq n$

(ii) up to reordering,

$(v_1, \ldots, v_m, w_{m+1}, \ldots, w_n)$ spans V.

Cor (Dimension) V be a finite dimensional vector space over F then: any two basis of V have the same number of vectors called the dimension of V, $\dim_F V$.

proof $(v_1, \dots, v_n)$, $(w_1, \dots, w_m)$ basis of V over F. Then:

- $(v_i)_{1 \le i \le n}$ free, $(w_i)_{1 \le i \le m}$ generating $\Longrightarrow n \le m$
- $(w_i)_{1 \le i \le m}$ free, $(v_i)_{1 \le i \le n}$ —— $\Longrightarrow m \le n$

$\square$

Cor Let V be an F vector space with finite dimension n. Then:

(i) any independent set of vectors has at most n elements, with equality iff it is a basis

(ii) any spanning set of vectors has at least

n elements, with equality iff it is a basis.

proof $\rightsquigarrow$ exercise.

Prop Let U, W be subspaces of V.
If U and W are finite dimensional, then so is $U+W$ and:

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W).$$

proof Pick a basis $v_1, \dots, v_\ell$ of $U \cap W$.
Extend to a basis:

$v_1, \dots, v_\ell, u_1, \dots, u_m$ of U

$v_1, \dots, v_\ell, w_1, \dots, w_n$ of W

<u>Claim</u> $(v_1, \dots, v_\ell, u_1, \dots, u_m, w_1, \dots, w_n)$ is a basis of $U+W$.

- generating family of $U+W$: obvious.
- free family (linearily independent).

$$\sum_{i=1}^{\ell} \alpha_i v_i + \sum_{i=1}^{m} \beta_i u_i + \sum_{i=1}^{n} \gamma_i w_i = 0$$

$$\Rightarrow \underbrace{\sum_{i=1}^{\ell} \alpha_i v_i + \sum_{i=1}^{m} \beta_i u_i}_{\in U} = \underbrace{-\sum_{i=1}^{n} \gamma_i w_i}_{\in W} \qquad (*)$$

$$\Rightarrow \sum_{i=1}^{n} \gamma_i w_i \in U \cap W \Rightarrow \sum_{i=1}^{\ell} \delta_i v_i = \sum_{i=1}^{n} \gamma_i w_i$$

$(v_1, \dots, v_\ell)$ basis of $U \cap W$

$$\Rightarrow \sum_{i=1}^{\ell} (\alpha_i - \delta_i) v_i + \sum_{i=1}^{m} \beta_i u_i = 0$$

$\Longrightarrow$ $(v_1, \dots, v_\ell, u_1, \dots, u_m)$ free

$\alpha_i = \delta_i, \quad \beta_i = 0$

$$\Rightarrow \sum_{i=1}^{\ell} \alpha_i v_i + \sum_{i=1}^{n} \gamma_i w_i = 0$$

$(v_1, \dots, v_\ell, w_1, \dots, w_n)$ free

$$\Rightarrow \alpha_i = \gamma_i = 0 \Rightarrow \alpha_i = \beta_i = \gamma_i = 0$$

$\square$

Prop If V is a finite dimensional vector space over F and $U \leq V$ (subspace) then U and V/U are also finite dimensional and:

$$\dim V = \dim U + \dim V/U$$

proof Let $(u_1, \dots, u_\ell)$ be a basis of U, extend it to a basis:

$(u_1, \dots, u_\ell, w_{\ell+1}, \dots, w_n)$ of V.

Claim $(w_{\ell+1} + U, \dots, w_n + U)$ basis of V/U. $\Leftarrow$ Exercise

$\blacksquare$

Rmk V vector space over F
$U \leq V$

We say that U is a proper subspace if $U \neq V$.

Then: U proper $\Rightarrow$ $\dim U < \dim V$.

$\left(V/U \neq \{0\} \Rightarrow \dim V/U > 0 \right.$
$\left. \Rightarrow \dim U < \dim V. \right)$

Def (Direct sum)

V vector space over F.
$U, W \leq V$ (subspaces)

We say that: $V = U \oplus W$
("V is the direct sum of U and W")
iff every element $v \in V$ can be written:
$v = u + w$ with $(u, w) \in U \times W$

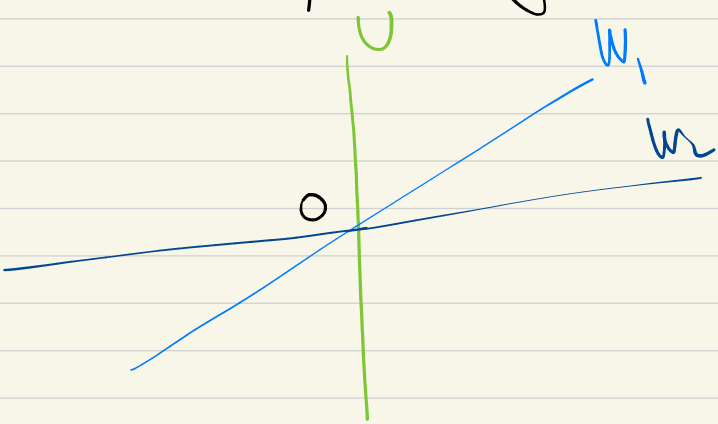
and this decomposition is unique.

Equivalently:

$$V = U \oplus W$$

$$\iff \forall v \in V, \ \exists! (u, w) \in U \times W \ / \ v = u + w.$$

⚠ We say that W is **a** direct complement of U in V. There is no uniqueness of such a complement.



$V = \mathbb{R}^2$

$V = U \oplus W_1$

$V = U \oplus W_2$

Notation We will systematically use the following notation: let two families of vectors

$$\mathcal{B}_1 = \{u_1, \dots, u_\ell\}$$
$$\mathcal{B}_2 = \{w_1, \dots, w_m\}$$

then: $\mathcal{B}_1 \cup \mathcal{B}_2 = \{v_1, \dots, v_\ell, w_1, \dots, w_m\}$ regardless of possible redundancies. In particular:

$$\{v_1\} \cup \{v_1\} = \{v_1, v_1\}$$

and hence for example the family $\{v_1, v_1\}$ can never be free.

Lemma $U, W \leq V$, then:

The Following Are Equivalent:

(i) $V = U \oplus W$

(ii) $V = U + W$ and $U \cap W = \{0\}$.

(iii) For any basis B_1 of U, B_2 of W, the union $B = B_1 \cup B_2$ is a basis of V

proof (ii) $\Rightarrow$ (i) $V = U + W \Rightarrow$

$\forall v \in V, \quad \exists (u, w) \in U \times W \ / \ v = u + w$.

Uniqueness $\quad u_1 + w_1 = u_2 + w_2 = v$

$$\Rightarrow \quad \underbrace{u_1 - u_2}_{\in U} = \underbrace{w_2 - w_1}_{\in W}$$

$$\Rightarrow \quad u_1 = u_2 \text{ and } w_1 = w_2,$$

$U \cap W = \{0\}$

(i) $\Rightarrow$ (iii) $\quad \mathcal{B}_1$ basis of U

$\mathcal{B}_2$ ______ W

$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$

- generating family of $U+W$ obvious
- $\mathcal{B}$ free family: $\quad \sum' \lambda_i v_i = 0 = 0_U + 0_W$

$$\underbrace{\sum_{u \in \mathcal{B}_1} \lambda_u u}_{U} + \underbrace{\sum_{w \in \mathcal{B}_2} \lambda_w w}_{W} = 0$$

$$\Longrightarrow \qquad \sum_{u \in \underline{\mathcal{B}_1}} \lambda_u u = \sum_{u \in \underline{\mathcal{B}_2}} \lambda_w w = 0$$

$\underset{\in U}{0} + \underset{\in W}{0} = 0$ $\qquad$ basis $\qquad$ basis

+ uniqueness

$\Longrightarrow \qquad \lambda_u = 0, \quad \lambda_w = 0$

$\mathcal{B}_1$ basis

$\mathcal{B}_2$ basis $\qquad \Rightarrow \mathcal{B}$ free family.

(iii) $\Rightarrow$ (ii) I need to show that:

$$\begin{cases} V = U + W \\ U \cap W = \{0\} \end{cases}$$

$$\left.\begin{array}{l} B_1 \text{ basis of } U \\ B_2 \text{ basis of } W \end{array}\right| \Rightarrow B = B_1 \cup B_2 \text{ basis of } V.$$

$$\Rightarrow v \in V, \quad v = \underbrace{\sum_{u \in B_1} \lambda_u u}_{\in U} + \underbrace{\sum_{w \in B_2} \lambda_w w}_{\in W}$$

$$\Rightarrow V = U + W.$$

Let $v \in U \cap W$, then:

$$v = \sum_{u \in B_1} \lambda_u u = \sum_{w \in B_2} \lambda_w w$$

$$\Rightarrow \sum_{u \in B_1} \lambda_u u - \sum_{w \in B_2} \lambda_w w = 0$$

$$\underset{B_1 \cup B_2 \text{ free}}{\Longrightarrow} \lambda_u = \lambda_w = 0$$

Def V vector space over F

$V_1, \ldots, V_\ell \leq V$ (subspaces)

(i) $$\sum_{i=1}^{\ell} V_i = \{ v_1 + \ldots + v_\ell, \; v_j \in V_j, \; 1 \leq j \leq \ell \}$$

(ii) The sum is direct

$$\left(\sum_{i=1}^{\ell} V_i = \bigoplus_{i=1}^{\ell} V_i \right)$$

iff:

$$v_1 + \ldots + v_\ell = v_1' + \ldots + v_\ell'$$

$$\Rightarrow \quad v_1 = v_1', \ldots, v_\ell = v_\ell'.$$

Equivalently: $V = \bigoplus_{i=1}^{\ell} V_i$

$$\Leftrightarrow \forall v \in V, \exists! (v_1, \ldots, v_\ell) \in \prod_{i=1}^{\ell} V_i \; /$$

$$v = \sum_{i=1}^{\ell} v_i . \qquad \square$$

Exercise TFAE:

(i) $\sum_{i=1}^{\ell} V_i = \bigoplus_{i=1}^{\ell} V_i$ (sum is direct)

(ii) $\forall i$, $V_i \cap \left(\sum_{j \neq i} V_j \right) = \{0\}$,

(iii) For any basis B_i of V_i, $B = \bigcup_{i=1}^{\ell} B_i$ is a basis of $\sum_{i=1}^{\ell} V_i$.