Lecture 22

Orthogonal complement / Adjoint map

Def Suppose $V = U \oplus W$
(U a complement of W in V)

We define: $\pi : V \longrightarrow W$

$$v = u + w \longmapsto w$$

- π linear
- $\pi^2 = \pi$

) We say that π is the projection operator onto W.

Remark $Id - \pi$ = projection onto U.

We can be more explicit when $U = W^{\perp}$
(U is the orthogonal complement of W in V)

Lemma Let V be an inner product space, $W \leq V$ finite dim. subspace of V.
$(e_1, \dots, e_k)$ ORTHONORMAL BASIS OF W.

Then:

(a) $$\pi(v) = \sum_{i=1}^{k} \langle v, e_i \rangle e_i, \quad \forall v \in V$$

(b) $$\|v - \pi(v)\| \leq \|v - w\| \quad \forall v \in V, \ w \in W$$
with equality iff $w = \pi(v)$

(ie $\pi(v)$ is the closest point in W to v)

proof (a) $\pi(v) = \sum_{i=1}^{k} \langle v, e_i \rangle e_i$

$W = \mathrm{Span}\langle e_1, \dots, e_k \rangle$

$$v = v - \pi(v) + \underbrace{\pi(v)}_{\in W}$$

I claim: $v - \pi(v) \in W^{\perp}$

$\Leftrightarrow \forall w \in W, \ \langle v - \pi(v), w \rangle = 0$

$\Leftrightarrow \forall 1 \leq j \leq k, \ \langle v - \pi(v), e_j \rangle = 0$

$$\langle v - \pi(v), e_j \rangle = \langle v, e_j \rangle - \left\langle \sum_{i=1}^{k} \langle v, e_i \rangle e_i, e_j \right\rangle$$

$$= \langle v, e_j \rangle - \langle v, e_j \rangle =$$

$$= 0$$

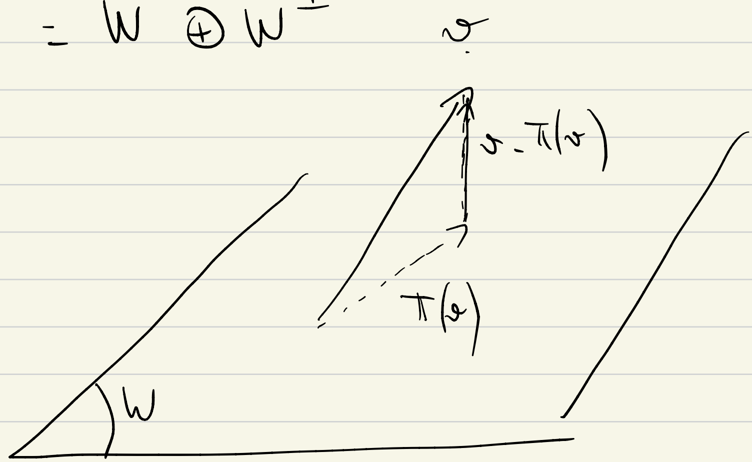
ORTHONORMAL

Indeed: $\quad v = \underbrace{v - \pi(v)}_{\in W^\perp} + \underbrace{\pi(v)}_{\in W}$

$\Rightarrow \ V = W + W^\perp \qquad (W \cap W^\perp = \{0\})$

$\Rightarrow \ V = W \overset{\perp}{\oplus} W^\perp$

(b)



We write: $w \in W$,

$$\|v - w\|^2 = \|\underbrace{v - \pi(v)}_{\in W^\perp} + \underbrace{\underbrace{\pi(v)}_{\in W} - \underbrace{w}_{\in W}}_{\in W}\|^2$$

$$= \|v - \pi(v)\|^2 + \|\pi(v) - w\|^2$$

$$\geqslant \|v - \pi(v)\|^2$$ with equality iff $w = \pi(v)$.

Adjoint map

Very fundamental object with in finite dimensional generalizations.

Def - Prop Let V, W be finite dimensional inner product spaces, $\alpha \in L(V, W)$.

Then there is a unique linear map

$$\alpha^* : W \longrightarrow V$$

such that: $\forall (v, w) \in V \times W,$

$$\langle \alpha(v), w \rangle = \langle v, \alpha^*(w) \rangle$$

<u>Moreover</u>, if:

$\mathcal{B}$ orthonormal basis of V
$\mathcal{C}$ ________________ of W ,

then:
$$[\alpha^*]_{\mathcal{C},\mathcal{B}} = \left(\overline{[\alpha]_{\mathcal{B},\mathcal{C}}}\right)^T$$

<u>proof</u> Brute force computation.

$$\mathcal{B} = \{v_1, \dots, v_n\}$$
$$\mathcal{C} = \{w_1, \dots, w_m\}$$
$$A = [\alpha]_{\mathcal{B}\,\mathcal{C}} = (a_{ij})$$

. **Existence**

$$[\alpha^*]_{\mathcal{C},\mathcal{B}} = \bar{A}^T = C = (c_{ij})$$

$$c_{ij} = \overline{a_{ji}}$$

$$\langle \alpha\left(\sum \lambda_i v_i\right), \sum c_j w_j \rangle$$

$$= \langle \sum_{i,k} \lambda_i a_{ki} w_k, \sum c_j w_j \rangle$$

$$= \sum_{i,j} \lambda_i a_{ji} \overline{c_j} \quad (*)$$

$\uparrow$ $(w_i)_i$ ORTHONORMAL

$$\langle \sum_i \lambda_i v_i, \alpha^*\left(\sum_j c_j w_j\right) \rangle$$

$$= \langle \sum_i \lambda_i v_i, \sum_{j,k} c_j c_{kj} v_k \rangle$$

$$= \sum_{i,j} \lambda_i \overline{c_{ij}} \overline{c_j} \quad (*)$$

$\uparrow$ $(v_i)_i$ ORTHONORMAL

$a_{ji} = \overline{c_{ij}}$

. <u>Uniqueness</u> Follows by using the above computation and computing $\alpha^*(\omega_j)$.

D.

<u>Notation</u> We will note: $A^+ = \bar{A}^T$

<u>Remark</u> We are using the same notation α^* for the adjoint and the dual of α.

V, W are real product inner spaces

$\alpha \in L(V, W)$

$$\psi_{R,V} : V \xrightarrow{\simeq} V^*$$
$$v \longmapsto \langle -, v \rangle$$

$$\psi_{R,W} : W \longrightarrow W^*$$
$$w \longmapsto \langle -, w \rangle$$

then the adjoint of α is given by:

$$W \xrightarrow[\psi_{\mathbb{R},W}]{} W^* \xrightarrow[\text{dual of } \alpha]{} V^* \xrightarrow[\psi_{\mathbb{R},V}^{-1}]{} V$$

(check as an exercise)

Self adjoint maps and Isometries.

Prop. / Def V inner product space, $\alpha \in L(V)$.
Let $\alpha^* \in L(V)$ be the adjoint map.

Condition	Equivalent	Name
$\langle \alpha v, w \rangle = \langle v, \alpha w \rangle$ $\forall (v,w)$ $\Longleftrightarrow$	$\alpha = \alpha^*$	self adjoint: • $\mathbb{R}$ symmetric • $\mathbb{C}$ Hermitian

Condition	Equivalent	Name
$\langle \alpha v, \alpha w\rangle = \langle v, w\rangle$ $\forall v, w$ $\Longleftrightarrow$	$\alpha^* = \alpha^{-1}$	Isometry: . $\mathbb{R}$: orthogonal . $\mathbb{C}$: unitary

proof Check the equivalence for isometries:

$$\langle \alpha v, \alpha w\rangle = \langle v, w\rangle \quad \forall v, w$$

$$\Longleftrightarrow \quad \alpha^* = \alpha^{-1}.$$

$\boxed{\Rightarrow}$ $v = w \Rightarrow \|\alpha v\|^2 = \|v\|^2$

$\Rightarrow \ker \alpha = \{0\} \Rightarrow \alpha$ injective

$\Rightarrow$ α bijective, α^{-1} well defined.

V fin dim

<u>Then</u> : $\forall (v, w) \in V \times V$,

$$\langle v, \alpha^* w \rangle \underset{\text{def } \alpha^*}{=} \langle \alpha v, w \rangle = \langle \alpha v, \alpha(\alpha^{-1} w) \rangle$$

$$\underset{(H)}{=} \langle v, \alpha^{-1} w \rangle$$

$\Rightarrow \forall v, \ \langle v, (\alpha^* - \alpha^{-1}) w \rangle = 0$

$v = (\alpha^* - \alpha^{-1}) w \Rightarrow \forall w,$

$$(\alpha^* - \alpha^{-1}) w = 0 \Rightarrow \alpha^* = \alpha^{-1}$$

<u>$\Leftarrow$)</u> $\langle \alpha v, \alpha w \rangle \underset{\text{def } \alpha^*}{=} \langle v, \alpha^* \alpha w \rangle \underset{\alpha^* \alpha = \mathrm{Id}}{=} \langle v, w \rangle$

Remark By the polarization identity,

α isometry $\iff$ $\underbrace{\forall v \in V, \quad \|\alpha(v)\| = \|v\|}_{\alpha \text{ preserves the norm.}}$

$(\alpha^* = \alpha^{-1})$

(Check as an exercise)

Lemma V fin dim real (complex) inner product space. Then $\alpha \in L(V)$

(i) is self adjoint iff for <u>any</u> orthonormal basis B of V, $[\alpha]_B$ is symmetric (Hermitian)

(ii) is an isometry iff for <u>any</u> orthonormal basis B of V,

$$\boxed{[\alpha]_B \text{ is orthogonal (unitary)}.}$$

proof B orthonormal basis, $[\alpha^*]_B = \overline{[\alpha]}_B^T$

. self adjoint : $\overline{[\alpha]}_B^T = [\alpha]_B$

. isometry : $\overline{[\alpha]}_B^T = [\alpha]_B^{-1}$ D.

Def V finite dim. inner product space.

. $\underline{F = \mathbb{R}}$ $O(V) = \{\alpha \in L(V),\ \alpha \text{ isometry}\}$

$\equiv$ orthogonal group of V

. $\underline{F = \mathbb{C}}$ $U(V) = \{\alpha \in L(V),\ \alpha \text{ isometry}\}$

$\equiv$ unitary group of V.

Recall: V fin dim. inner product space
$\{e_1, \dots, e_n\}$ orthonormal basis

. $F = \mathbb{R}$

$$O(V) \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{orthonormal basis} \\ \text{of } V \end{array} \right\}$$

$$\alpha \longmapsto (\alpha(e_1), \dots, \alpha(e_n))$$

. $F = \mathbb{C}$

$$U(V) \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{orthonormal basis} \\ \text{of } V \end{array} \right\}$$

$$\alpha \longmapsto (\alpha(e_1), \dots, \alpha(e_n))$$