Lecture 21

Gram Schmidt and orthogonal complement

Def V vector space over $\mathbb{R}$ (or $\mathbb{C}$). An inner product is a positive definite symmetric (Hermitian) form φ on V.

Notation $\langle u, v \rangle = \varphi(u, v)$

Norm $\|v\| = \sqrt{\underbrace{\langle v, v \rangle}_{\in \mathbb{R}_+}}$

and: $\|v\| = 0 \iff \langle v, v \rangle = 0$

$\underset{\substack{\uparrow \\ \text{definite}}}{\iff} v = 0$

Lemma (Cauchy - Schwarz)

$$|\langle u, v\rangle| \leq \|u\| \, \|v\|$$

Proof $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $t \in \mathbb{F}$, then:

$$\underbrace{0 \leq}_{\text{positive}} \|tu - v\|^2 = \langle \underset{\substack{\uparrow \\ \text{linear}}}{tu - v}, \underset{\substack{\uparrow \\ \text{antilinear}}}{tu - v}\rangle$$

$$= t\bar{t}\langle u, u\rangle - t\langle u, v\rangle - \bar{t}\underbrace{\langle v, u\rangle}_{\overline{\langle u, v\rangle}} + \langle v, v\rangle$$

$$= |t|^2 \|u\|^2 - 2\,\mathrm{Re}\left(t\langle u, v\rangle\right) + \|v\|^2$$

We pick explicitly $t = \dfrac{\overline{\langle u, v\rangle}}{\|u\|^2}$ which gives:

$$0 \leq \frac{|\langle u, v\rangle|^2}{\|u\|^4}\|u\|^2 - 2\,\mathrm{Re}\left(\frac{|\langle u, v\rangle|^2}{\|u\|^2}\right) + \|v\|^2$$

$$\iff 0 \leq \|v\|^2 - \frac{|\langle u, v\rangle|^2}{\|u\|^2}$$

$$\iff |\langle u, v\rangle|^2 \leq \|u\|^2 \|v\|^2$$ D.

Exercise Show that if there is equality in Cauchy Schwarz, then u, v are colinear.

$$\boxed{\begin{array}{c}\textbf{Cor (Triangle inequality)} \\ \|u+v\| \leq \|u\| + \|v\|\end{array}}$$

proof

$$\|u+v\|^2 = \langle u+v, u+v\rangle$$

$$= \|u\|^2 + 2\operatorname{Re}(\langle u, v\rangle) + \|v\|^2$$

CS

$$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2$$ D.

Cor $\|.\|$ is a norm.

Def A set $(e_1, \dots, e_k)$ of vectors of V is

(i) ORTHOGONAL if $\langle e_i, e_j \rangle = 0$ for $i \neq j$

(ii) ORTHONORMAL if $\langle e_i, e_j \rangle = \delta_{ij}$

$$\left(\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \right)$$

Lemma If $(e_1, \dots, e_k)$ are orthogonal non zero vectors, then they are linearly independent.

Moreover, if:

$$v = \sum_{j=1}^{k} \lambda_j e_j, \quad \text{then:}$$

$$\lambda_j = \frac{\langle v, e_j \rangle}{\|e_j\|^2}$$

proof (i) $\sum_{i=1}^{k} \lambda_i e_i = 0$

$$\Rightarrow 0 = \langle \sum_{i=1}^{k} \lambda_i e_i , e_j \rangle$$

$$= \sum_{i=1}^{k} \lambda_i \langle e_i , e_j \rangle = \lambda_j \underbrace{\|e_j\|^2}_{\neq 0}$$

$$\Rightarrow \forall\, 1 \le j \le k ,\ \lambda_j = 0 .$$

(ii) $$v = \sum_{i=1}^{k} \lambda_i e_i$$

$$\Rightarrow \langle v , e_j \rangle = \lambda_j \|e_j\|^2$$

$$\Rightarrow \lambda_j = \frac{\langle v , e_j \rangle}{\|e_j\|^2}$$ $\square$.

Lemma (Parseval's identity)

If V is a finite dimensional inner product space and $(e_1, \dots, e_n)$ is an

ORTHONORMAL basis. Then:

$$\boxed{\langle u, v \rangle = \sum_{i=1}^{n} \langle u, e_i \rangle \overline{\langle v, e_i \rangle}}$$

proof

$$u = \sum_{i=1}^{n} \langle u, e_i \rangle e_i \qquad (\|e_i\| = 1)$$

$$v = \sum_{i=1}^{n} \langle v, e_i \rangle e_i$$

$$\Rightarrow \langle u, v \rangle = \left\langle \sum_{i=1}^{n} \langle u, e_i \rangle e_i, \sum_{i=1}^{n} \langle v, e_i \rangle e_i \right\rangle$$

$$= \sum_{i=1}^{n} \langle u, e_i \rangle \overline{\langle v, e_i \rangle}$$

D.

In particular, in <u>an orthonormal basis</u>,

$$\|v\|^2 = \langle v, v \rangle = \sum_{i=1}^{n} |\langle v, e_i \rangle|^2$$

Theorem (GRAM_SCHMIDT orthogonalisation process)

V inner product space, $(v_i)_{i \in I}$, $v_i \in V$ / $(v_i)_{i \in I}$ linearily independent / I countable (or finite)

Then there exist a sequence $(e_i)_{i \in I}$ of ORTHONORMAL vectors such that:

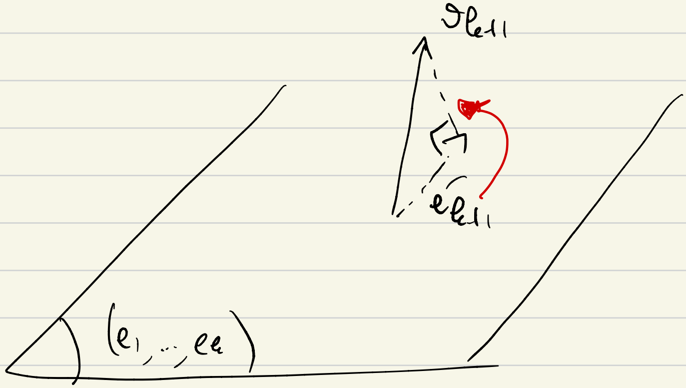
$$\operatorname{Span}\langle v_1, \dots, v_k \rangle = \operatorname{Span}\langle e_1, \dots, e_k \rangle \quad \forall k \geq 1$$

proof . Induction on k

. $k = 1$, $e_1 = \dfrac{v_1}{\|v_1\|}$

. Say we found $(e_1, \dots, e_k)$. We look for e_{k+1}. Define:

$$e'_{k+1} = v_{k+1} - \sum_{i=1}^{k} \langle v_{k+1}, e_i \rangle e_i$$



- $e'_{k+1} \neq 0$: otherwise, $v_{k+1} \in \mathrm{Span}\langle e_1, \dots, e_k \rangle$
  $= \mathrm{Span}\langle v_1, \dots, v_k \rangle$ contradiction

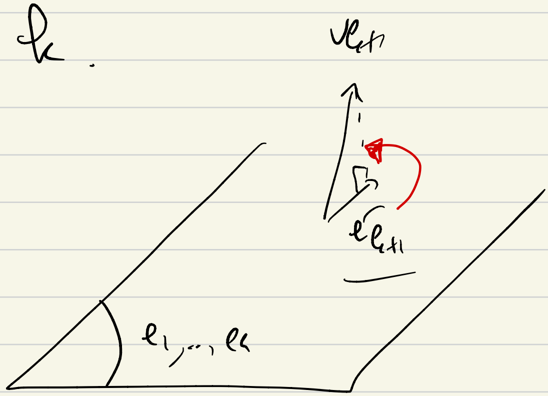
- $\langle e'_{k+1}, e_j \rangle = \langle v_{k+1}, e_j \rangle - \langle v_{k+1}, e_j \rangle = 0$
  $1 \leq j \leq k$

- $\mathrm{Span}\langle v_1, \dots, v_{k+1} \rangle = \mathrm{Span}\langle e_1, \dots, e_k, e'_{k+1} \rangle$

$\rightarrow$ $e_{k+1} = \dfrac{e'_{k+1}}{\|e'_{k+1}\|}$ does the job

D.

$\Rightarrow$ This is an algorithm to compute $(e_1, \dots, e_k)$ for all k.



Corollary V finite dim. inner product space. Any orthonormal set of vectors can be extended to an orthonormal basis of V

proof Pick $(e_1, \dots, e_k)$ orthonormal. Then they are linearly independent, so we can extend $(e_1, \dots, e_k, v_{k+1}, \dots, v_n)$ basis of V. Apply Gram Schmidt to this set (observe that there is no need to modify $e_1, \dots, e_k$)

$\Rightarrow (e_1, \dots, e_k, e_{k+1}, \dots, e_n)$ orthonormal basis of V.

Note $A \in M_{m,n}(\mathbb{R})$ has ortho~~gonal~~normal
$(\mathbb{C})$

columns if: $A^T A = Id \quad (\mathbb{R})$

$A^T \bar{A} = Id \quad (\mathbb{C})$

Def $A \in M_n(\mathbb{R})$ is:

$(\mathbb{R})$ - orthogonal if: $A^T A = Id \quad (\Leftrightarrow A^{-1} = A^T)$

$(\mathbb{C})$ - unitary if: $A^T \bar{A} = Id \quad (\Leftrightarrow A^{-1} = \bar{A}^T)$

Prop $A \in M_n(\mathbb{R})\ ([\mathbb{C}])$, non singular, then

A can be written: $A = R\,T$

where : . T upper triangular
. R orthogonal (unitary)

proof Exercise : apply Gram Schmidt to the column vectors of A. $\square$

Orthogonal complement and projection

($\rightsquigarrow$ natural and deep extension to the infinite dimensional setting : HILBERT spaces)

Def V inner product space, $V_1, V_2 \leq V$.
We say that V is the orthogonal direct sum of V_1 and V_2 if :

(i) $V = V_1 \oplus V_2$

(ii) $\forall (v_1, v_2) \in V_1 \times V_2 \quad \langle v_1, v_2 \rangle = 0$

<u>Notation</u> $V = V_1 \overset{\perp}{\oplus} V_2$

$\left(V = V_1 \textcircled{\perp} V_2 \right)$

<u>Rk</u> $v \in V_1 \cap V_2 \underset{(ii)}{\Rightarrow} \langle v, v \rangle = \|v\|^2 = 0$

$\Rightarrow v = 0$.

<u>Def</u> V inner product space

$W \leq V$. We define:

$$W^{\perp} = \{ v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W \}$$

<u>Lemma</u> V inner product space, $\dim V < +\infty$,

$W \leq V$. Then:

$$V = W \overset{\perp}{\oplus} W^{\perp}$$

$(*)$

Proof Observe that $W^\perp \leq V$.

(ii) $\begin{cases} w \in W \\ w^\perp \in W^\perp \end{cases}$,

$\langle w, w^\perp \rangle = 0$ by definition of $W^\perp$.

Moreover $w \in W \cap W^\perp \Rightarrow \|w\|^2 = \langle w, w \rangle = 0$

$\Rightarrow w = 0$.

(i) We need to show that $V = W + W^\perp$

Let $(e_1, \dots, e_k)$ be an orthonormal basis of W.
Extend it to $(e_1, \dots, e_k, e_{k+1}, \dots, e_n)$
orthonormal basis of V. Observe that:

$(e_{k+1}, \dots, e_n) \in W^\perp$

$\Rightarrow V = W + W^\perp$