Lecture 20

Hermitian form / $\mathbb{C}$
Skew symmetric forms / $\mathbb{R}$

φ sesquilinear $\quad V \times V \to \mathbb{C}$

$\dim_{\mathbb{C}} V$

. linear first variable $\quad \varphi(\lambda u, v) = \lambda \varphi(u, v)$

. antilinear second variable $\quad \varphi(u, \lambda v) = \bar{\lambda} \varphi(u, v)$

$\boxed{\text{Def}}$ (Hermitian form) A sesquilinear form $\varphi : V \times V \to \mathbb{C}$ is called Hermitian if :

$$\forall (u, v) \in U \times V, \quad \varphi(u, v) = \overline{\varphi(v, u)}$$

Remark φ Hermitian, $\varphi(u,u) = \overline{\varphi(u,u)}$

$\Rightarrow \varphi(u,u) \in \mathbb{R}$

Moreover: $\varphi(\lambda u, \lambda u) = |\lambda|^2 \varphi(u,u)$

$\rightsquigarrow$ We may talk about positive / negative (semi) definite Hermitian form.

Lemma A sesquilinear form $\varphi : V \times V \longrightarrow \mathbb{C}$ is Hermitian iff: for any basis $\mathcal{B}$ of V,

$$[\varphi]_{\mathcal{B}} = \overline{[\varphi]_{\mathcal{B}}^{T}}$$

proof

$$A = [\varphi]_{\mathcal{B}} = (a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}}$$

$$a_{ij} = \varphi(e_i, e_j)$$

So :

$$\begin{cases} a_{ij} = \varphi(e_i, e_j) \\ a_{ji} = \varphi(e_j, e_i) = \overline{\varphi(e_i, e_j)} = \overline{a_{ij}} \end{cases}$$

φ Hermitian

$$\Longrightarrow \quad [\varphi]_B^T = \overline{[\varphi]_B}$$

<u>Conversely</u> $\quad [\varphi]_B = A, \quad A = \overline{A}^T$

$$\begin{cases} u = \sum_{i=1}^n \lambda_i e_i \\ v = \sum_{i=1}^n \mu_i e_i \end{cases} \qquad B = (e_1, \dots, e_n)$$

$$\varphi(u, v) = \varphi\left(\sum_{i=1}^n \lambda_i e_i, \sum_{i=1}^n \mu_i e_i\right)$$

$$= \sum_{i,j=1}^n \lambda_i \overline{\mu_j} \varphi(e_i, e_j)$$

$$= \sum_{i,j=1}^n \lambda_i \overline{\mu_j} a_{ij}$$

$$\overline{\varphi(v,u)} = \overline{\varphi\left(\sum_{i=1}^{n} \mu_i e_i, \sum_{i=1}^{n} \lambda_i e_i\right)}$$

$$= \overline{\sum_{i,j=1}^{n} \mu_i \overline{\lambda_j}\, \varphi(e_i, e_j)}$$

$$= \sum_{i,j=1}^{n} \overline{\mu_i}\, \lambda_j\, \overline{a_{ij}} = \sum_{i,j=1}^{n} \overline{\mu_j}\, \lambda_i\, \underbrace{\overline{a_{ji}}}_{= a_{ij}}$$

$$= \sum_{i,j=1}^{n} \lambda_i \overline{\mu_j}\, a_{ij} = \varphi(u,v) \qquad \square$$

Polarisation identity A Hermitian form φ on a complex vector space V is entirely determined by:

$$Q : V \to \mathbb{R}$$
$$v \mapsto \varphi(v,v)$$

via the formula:

$$\varphi(u,v) = \frac{1}{4}\Big[Q(u+v) - Q(u-v) + iQ(u+iv) - iQ(u-iv)\Big]$$

→ Exercice Check the polarisation identity.

Thm (Hermitian formulation of Sylvester's law)

V be a finite dim. vector space over $\mathbb{C}$.
$\dim_{\mathbb{C}} V = n$.

Let $\varphi : V \times V \to \mathbb{C}$ be a Hermitian form on V.
Then $\exists\, \mathcal{B} = (v_1, \dots, v_n)$ of V s.t.:

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} I_p & & 0 \\ & -I_q & \\ 0 & & 0 \end{pmatrix}$$

$$\boxed{\text{where } p, q \text{ depend only on } \varphi.}$$

proof (sketch: nearly identical to the real case)

. Existence $\varphi = 0$, done. Assume not, then by the polarization identity there exist $e_1 \neq 0$ such that $\varphi(e_1, e_1) \neq 0$.

Rescale to: $v_1 = \dfrac{e_1}{\sqrt{|\varphi(e_1, e_1)|}}$

so:
$$\underbrace{\varphi(v_1, v_1)}_{\in \mathbb{R}} = \pm 1.$$

Then we consider the orthogonal:

$$W = \{ w \in V \ / \ \varphi(v_1, w) = 0 \}$$

and we check (verbatim like in the real case):

$$\boxed{V = \langle v_1 \rangle \oplus W} \qquad (\dim W = n-1)$$

$\Rightarrow$ Now argue by induction on the dimension to diagonalize $\varphi|_W$

. Uniqueness of p: p is the maximal dimension of a subspace on which φ is definite positive ($\varphi(u,u) \in \mathbb{R}$)

(verbatim like in the real case)

. Similarly for q.

□.

$\boxed{F = \mathbb{R}}$ V v. space over $\mathbb{R}$.

Def A bilinear form on a real vector space V is <u>skew symmetric</u> if :

$$\varphi(u, v) = -\varphi(v, u) \quad \forall u, v \in V \times V .$$

<u>Rk</u> (i) $\varphi(u, u) = -\varphi(u, u)$

$$\Rightarrow \quad \varphi(u, u) = 0$$

(ii) $\forall B$ basis of V,

$$[\varphi]_B = -[\varphi]_B^T$$

(iii) $\forall A \in M_n(\mathbb{R})$,

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew symmetric}}$$

Thm (Sylvester form) Let φ be a skew symmetric bilinear form over V (vector space over $\mathbb{R}$), then: $\exists$ a basis

$\dim_{\mathbb{R}} V = n < +\infty$

$$\mathcal{B} = \left(v_1, w_1, v_2, w_2, \dots, v_m, w_m, v_{2m+1}, v_{2m+2}, \dots, v_n\right)$$

s.t.:

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & & & & & & & \\ -1 & 0 & & & & & & 0 & \\ & & 0 & 1 & & & & & \\ & & -1 & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & 1 & & \\ & 0 & & & & -1 & 0 & & \\ & & & & & & & 0 & \end{pmatrix}$$

Cor Skew symmetric matrices have an even rank

proof (sketch) Induction on the dimension of V.

- $\varphi \equiv 0 \rightarrow$ done
- $\varphi \neq 0 \Rightarrow \exists\, v_1, w_1 \,/\, \varphi(v_1, w_1) \neq 0$

- $v_1, w_1 \neq 0 \Rightarrow$ after scaling w_1 we can assume $\varphi(v_1, w_1) = 1$

$$\Rightarrow \quad \varphi(w_1, v_1) = -1$$

- v_1, w_1 linearily independent

$$\varphi(v_1, \lambda v_1) = \lambda \varphi(v_1, v_1) = 0$$

- $U = \langle v_1, w_1 \rangle$

$$W = \{ v \in V \,/\, \varphi(v_1, v) = \varphi(w_1, v) = 0 \}$$

Exercise $\boxed{V = U \oplus W}$

Now apply the induction hypothesis to $\varphi|_W$ □.

Inner product spaces

- definite positive bilinear forms

$\longrightarrow$ scalar product

$\longrightarrow$ norms : notion of distance

- $\Longrightarrow$ spectacular infinite dimensional counterpart

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Def (Inner product) Let V be a vector space over $\mathbb{R}$ (resp. $\mathbb{C}$). An inner product on V is a positive definite symmetric (resp. Hermitian) form φ on V.

<u>Notation</u> $\langle u, v \rangle = \varphi(u, v)$

V is called a real (resp. complex) inner product space.

Examples

(i) $\mathbb{R}^n$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

(ii) $\mathbb{C}^n$, $\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$

(iii) $V = \mathcal{C}([0,1], \mathbb{C})$,

$$\langle f, g \rangle \overset{\text{def}}{=} \int_0^1 f(t)\,\overline{g(t)}\,dt$$

"L^2 scalar product"

(iv) We can fix a weight $\omega : [0,1] \to \mathbb{R}_+^*$

and define on $V = \mathcal{C}([0,1], \mathbb{C})$:

$$\langle f, g \rangle = \int_0^1 f(t)\, \bar{g}(t)\, \omega(t)\, dt .$$

$\Longrightarrow$ One can check that these are inner products.

$$\Big(\langle u, u \rangle = 0 \implies u = 0 \Big) \quad (\text{definite})$$

$\boxed{\text{Def}}$ (Norm / length)

$$\|v\| = \left(\langle v, v \rangle \right)^{\frac{1}{2}}$$

norm derives from a scalar product

<u>Rk</u> $\langle v, v \rangle \in \mathbb{R}_+$, and

$$\|v\| = 0 \implies v = 0$$

positive <u>definite</u>

11.11 : this allows us to define a notion of length.