Lecture 2 Spans, linear independance and STEINITZ exchange lemma

$\rightsquigarrow$ dimension of a vector space
basis.

Def (Span of a family of vectors)

Let V be an F-vector space. Let $S \subset V$ be a subset (S = set of vectors).

We define:

$$\langle S \rangle = \left\{ \text{finite linear combinations of elements of } S \right\}$$

$\langle S \rangle$ = span of S

$$= \left\{ \sum_{s \in S}' \underset{\in F}{\lambda_s} \underset{\in V}{v_s}, \ v_s \in S, \text{ only finitely} \right.$$

$\boxed{\text{many } \lambda\text{s are non zero}\}}$

Convention $\langle \emptyset \rangle = \{0\}$

Remark $\langle S \rangle$ = smallest vector subspace of V which contains S.

Ex ① $V = \mathbb{R}^3$

$$S = \left\{ \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}, \begin{vmatrix} 0 \\ 1 \\ 2 \end{vmatrix}, \begin{vmatrix} 3 \\ -2 \\ -4 \end{vmatrix} \right\}$$

$$\Rightarrow \quad \langle S \rangle = \left\{ \begin{vmatrix} a \\ b \\ 2b \end{vmatrix}, (a, b) \in \mathbb{R} \right\}$$

② $V = \mathbb{R}^n = \left\{ \begin{vmatrix} x_1 \\ \vdots \\ x_n \end{vmatrix}, x_i \in \mathbb{R}, 1 \leq i \leq n \right\}$

$$e_i = \begin{vmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{vmatrix} \begin{matrix} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{matrix} \qquad V = \operatorname{Span}(e_i)_{1 \leq i \leq n}$$

③ . X set , $V = \mathbb{R}^X = \{ f : X \to \mathbb{R} \}$

. $\delta_x : X \to \mathbb{R}$ $\quad (x \in X)$

$$y \mapsto \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

. $\mathrm{Span}\,(\delta_x)_{x \in X}$

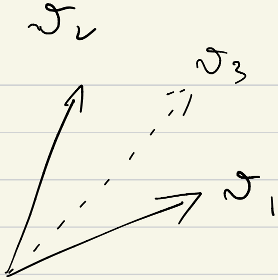
$= \{ f \in \mathbb{R}^X \,/\, f \text{ has finite support} \}$

$\left(\mathrm{Supp} f = \{ x \,/\, f(x) \neq 0 \} \right)$

Def Let V be an F-vector space. Let S be a subset of V. We say that S spans V if :

$$\langle S \rangle = V$$

Ex $V = \mathbb{R}^2$



$$\langle \{v_1; v_2\} \rangle_{\mathbb{R}^2} = \mathbb{R}^2$$
$$\langle \{v_1; v_2, v_3\} \rangle_{\mathbb{R}^2} = \mathbb{R}^3$$

Def (Finite dimension)
Let V be an F-vector space. We say that V is finite dimensional if it is spanned by a finite set.

Example
- $V = \mathbb{P}[x]$: polynomials on $\mathbb{R}$
- $V_n = \mathbb{P}_n[x]$: __________ with degree $\leq n$, $n \in \mathbb{N}$.

. $V_n = \langle \{1; x; \dots; x^n\} \rangle$

$\Rightarrow$ V_n is finite dimensional

Exercise: $V = \mathbb{R}[x]$ is infinite dimensional

($\equiv$ not finite dimensional: there is no family S with finitely many elements which span is V).

$\rightsquigarrow$ If V is finite dimensional, is there a minimal number of vectors in the family required so that the family spans V ?

Def (Independance) We say that $(v_1, \dots, v_n)$ elements of V are LINEARILY INDEPENDENT if :

$$\left.\begin{array}{l}\sum_{i=1}^{n} \lambda_i v_i = 0 \\ \lambda_i \in F\end{array}\right| \Longrightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$$

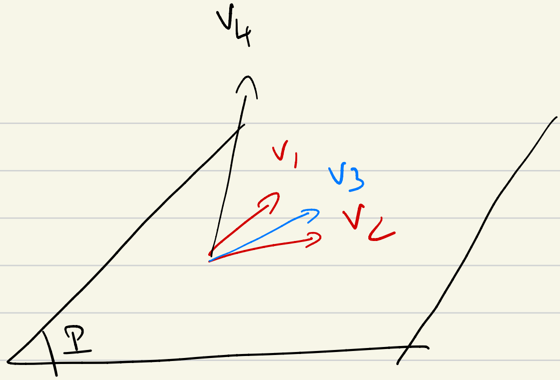
Equivalently, $(v_1, \dots, v_n)$ are not linearly independent if one of them is a linear combination of the $(n-1)$ remaining ones.

Indeed, $\exists (\lambda_1, \dots, \lambda_n)$, $j \in [\![1, n]\!]$ /

$$\left.\begin{array}{l}\sum_{i=1}^{n} \lambda_i v_i = 0 \\ \lambda_j \neq 0\end{array}\right|$$

$$\Longrightarrow \quad v_j = -\frac{1}{\lambda_j} \sum_{\substack{i \neq j \\ 1 \leq i \leq n}} \lambda_i v_i \quad .$$

Ex $\mathbb{R}^3$



- (v_1, v_2) independent
- $v_3 \in \langle v_1, v_2 \rangle \Rightarrow (v_1, v_2, v_3)$ NOT linearly independent
- $v_4 \notin \langle v_1, v_2 \rangle \Rightarrow (v_1, v_2, v_4)$ are linearly independent.

Rk $(v_i)_{1 \leq i \leq n}$ linearly independent

$\Rightarrow \forall i \in [\![1;n]\!], \quad v_i \neq 0.$

Def (Basis) A subset S of V is a BASIS of V if:

(i) $\langle S \rangle = V$

(ii) S linearly independent

Rk When S spans V, we say that S is a generating family. So a basis is a linearly independent generating family.

Ex 1 $V = \mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, x_i \in \mathbb{R}, 1 \leq i \leq n \right\}$

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{ith position} \\ \\ \\ \\ \end{matrix}$$

$(e_i)_{1 \leq i \leq n}$ basis for V.

Exercise.

2 $V = \mathbb{C}$

. $\mathbb{C} \equiv \mathbb{C}\,(= F)$ vector space, $\{1\}$ a basis

. $\mathbb{C} = \mathbb{R}\,(= F)$ ______ , $\{1, i\}$ a basis

3 $V = \mathbb{R}[x] = \{ P(x)$ polynomials on $\mathbb{R} \}$

$S = \{ x^n, \ n \geq 0 \} \equiv$ basis of V.

<u>Lemma</u> V F vector space. Then $(v_1, \dots, v_n)$ is a basis of V if and only if (iff) any vector $v \in V$ has a unique decomposition:

$$v = \sum_{i=1}^{n} \lambda_i v_i, \quad \lambda_i \in F$$

<u>Rmk</u> $(\lambda_1, \dots, \lambda_n) \equiv$ coordinate of v in the basis $(v_1, \dots, v_n)$.

proof. $\langle v_1, \dots, v_n \rangle = V$

$\Longrightarrow \forall v \in V, \exists (\lambda_1, \dots, \lambda_n) \in F^n \;/$

$$v = \sum_{i=1}^{n} \lambda_i v_i$$

. $v = \sum_{i=1}^{n} \lambda_i v_i = \sum_{i=1}^{n} \lambda_i' v_i$

$\Longrightarrow \quad \sum_{i=1}^{n} (\lambda_i - \lambda_i') v_i = 0$

$\Longrightarrow \quad \lambda_i = \lambda_i', \ \forall 1 \leq i \leq n$

$(v_i)_{1 \leq i \leq n}$ linearly independent

Lemma If $(v_1, \dots, v_n)$ spans V, then some subset of this family is a basis of V

proof If $(v_1, \dots, v_n)$ are linearly independent, done. If they are not, then up to changing indices,

$$v_n \in \mathrm{Span}(v_1, \dots, v_{n-1})$$

$(v_n$ is a linear combination of $v_1, \dots, v_{n-1})$

$$\Rightarrow \langle v_1, \dots, v_n \rangle = \langle v_1, \dots, v_{n-1} \rangle$$

$$\begin{matrix} \parallel \\ V \end{matrix}$$

$$\Rightarrow \langle v_1, \dots, v_{n-1} \rangle = V$$

I iterate this process ◻

Th (STEINITZ EXCHANGE LEMMA)

Let V be a <u>finite</u> dimensional vector space over F. Take $(v_1, \dots, v_m)$ linearly independent, and $(w_1, \dots, w_n)$ which spans V. Then:

(i) $m \le n$

(ii) Up to reordering, $(v_1, \dots, v_m, w_{m+1}, \dots, w_n)$ spans V.

proof (Induction) Suppose that we have replaced $\ell\ (\geq 0)$ of the w_i. Reordering if necessary, $\boxed{\langle v_1, \dots, v_\ell, w_{\ell+1}, \dots, w_n \rangle = V}$.

- $m = \ell$ done
- Assume $\ell < m$. Then: $v_{\ell+1} \in V$

$$v_{\ell+1} = \sum_{i \leq \ell}' \alpha_i v_i + \sum_{i > \ell} \beta_i w_i$$

Since the $(v_i)_{1 \leq i \leq m}$ $(\ell+1 \leq m)$

linearly independent, then one of the β_i

is non zero. Up to reordering:

$$w_{\ell+1} = \frac{1}{\beta_{\ell+1}} \left(v_{\ell+1} - \sum_{i \leq \ell} \alpha_i v_i - \sum_{i > \ell+1} \beta_i w_i \right)$$

$\Rightarrow$ V is spanned by

$v_1, \ldots, v_{\ell+1}, w_{\ell+2}, \ldots, w_n$.

$\rightsquigarrow$ done after m steps

$\Rightarrow$ we must have replaced m w_i

$\Rightarrow$ $\boxed{m \leq n}$.