Lecture 18 — Bilinear forms

Bilinear form $\varphi : V \times V \to F$ bilinear form.

- $\dim_F V < +\infty$, $\mathcal{B}$ basis of V

- $[\varphi]_{\mathcal{B}} = [\varphi]_{\mathcal{B},\mathcal{B}} = (\varphi(e_i, e_j))_{1 \le i,j \le n}$

  $\mathcal{B} = (e_i, e_j)$

Lemma $\varphi : V \times V \to F$ bilinear

$\mathcal{B}, \mathcal{B}'$ basis for V, $P = [\mathrm{Id}]_{\mathcal{B}', \mathcal{B}}$

$$\Longrightarrow [\varphi]_{\mathcal{B}'} = P^T [\varphi]_{\mathcal{B}} P$$

proof Special case of general formula of Lecture 10. $\square$

Def $A, B \in M_n(F)$ are congruent if $\exists\, P$ invertible such that: $A = {}^tP B P$.

Rk This is an equivalence relation.

Def A bilinear form φ on V is symmetric if:

$$\varphi(u, v) = \varphi(v, u) \quad \forall u, v \in V$$

Remarks

- $A \in M_n(F)$, we say that
  A symmetric $\iff A = A^T$
  $\iff \left[A = (a_{ij})_{1 \leq i,j \leq n},\ a_{ij} = a_{ji}\right]$

- φ is symmetric $\iff [\varphi]_B$ is symmetric in ANY basis B.

. To be able to represent φ by a diagonal matrix, you must be symmetric:

$$P^T A P = \underset{\substack{\uparrow \\ \text{diagonal}}}{D} \Rightarrow \underset{\substack{\| \\ D}}{D^T} = P^T A^T P$$

$$\Rightarrow A = A^T$$

Def

A map $Q : V \longrightarrow F$ is a quadratic form iff there exist a bilinear form $\varphi : V \times V \longrightarrow F$ such that:

$$Q(u) = \varphi(u, u)$$

<u>Remark</u>

$\mathcal{B} = (e_i)_{1 \le i \le n}$

$A = [\varphi]_{\mathcal{B}} = (\underbrace{\varphi(e_i, e_j)}_{a_{ij}})_{1 \le i, j \le n}$

Then: $u = \sum_{i=1}^{n} x_i e_i$, $\mathrm{X}=X$, then:

$$Q(u) = \varphi\left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} x_i e_i\right)$$

$$= \sum_{i,j=1}^{n} x_i x_j \varphi(e_i, e_j)$$

$\uparrow$ φ bilinear

$$= \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

$$= X^T A X, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A = (a_{ij})$$

$\uparrow$ direct check

$$\begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \cdot$$

$$\boxed{Q(X) = X^T A X.}$$

. <u>Observe</u> $\quad x^T A x = \sum_{i,j=1}^{n} a_{ij} x_i x_j$

$$= \sum_{i,j=1}^{n} a_{ji} x_i x_j$$

$$= \frac{1}{2} \sum_{i,j=1}^{n} \left(a_{ij} + a_{ji}\right) x_i x_j$$

$$= \frac{1}{2} x^T \big(\underbrace{A + A^T}_{\text{symmetric}}\big) x$$

Prop If $Q : V \times V \to F$ is a quadratic form, then there exists a unique <u>symmetric</u> bilinear form $\varphi : V \times V \to F$ such that:

$$Q(u) = \varphi(u, u) \quad \forall u \in V$$

<u>proof</u> . Let φ bilinear form on V /

$\forall u,\ Q(u) = \psi(u,u)$. Let:

$$\varphi(u,v) = \frac{1}{2}\underbrace{\left(\psi(u,v) + \psi(v,u)\right)}, \quad \text{then:}$$

- φ symmetric
- $\varphi(u,u) = \psi(u,u) = Q(u)$

$\left(\rightsquigarrow \text{in a basis, } A \longrightarrow A + A^T\right)$

- <u>Uniqueness</u> φ is a symmetric bilinear form, then: $\forall (u,v) \in V$,

$$Q(u+v) = \varphi(u+v, u+v)$$
$$= \varphi(u,u) + \varphi(u,v) + \varphi(v,u) + \varphi(v,v)$$
$$= Q(u) + 2\varphi(u,v) + Q(v)$$

$$\Rightarrow \boxed{\varphi(u,v) = \frac{1}{2}\left[Q(u+v) - Q(u) - Q(v)\right]}$$

$\equiv$ POLARIZATION IDENTITY.

D.

Thm (Diagonalization of symmetric bilinear forms)

Let $\varphi : V \times V \longrightarrow F$ be a symmetric bilinear form, ($\dim_F V < +\infty$). Then there exist a basis of V such that:

$$[\varphi]_{\mathcal{B}} \text{ is diagonal.}$$

proof Induction on dimension.

- $n = 0, 1$ ✓.
- Suppose Thm holds for all dimension $< n$.
- If $\varphi(u,u) = 0 \ \forall u \in V \Rightarrow \varphi$ is identically zero

polarization identity

$\Rightarrow$ done.

$\cdot\ \varphi \neq 0$, I can always find $u \in V \setminus \{0\}$ /

$$\varphi(u, u) \neq 0 .$$

Let us call $u = e_1$. Let us define:

$$U = \left(\langle e_1 \rangle\right)^{\perp} \overset{\text{def}}{=} \{ v \in V \ / \ \varphi(e_1, v) = 0 \}$$

orthogonal

$$= \operatorname{Ker} \left\{ \begin{aligned} \varphi(e_1, \cdot) : V &\to F \\ v &\mapsto \varphi(e_1, v) \end{aligned} \right\}$$

Rank nullity $\quad \dim V = n = \underbrace{1}_{\operatorname{rk}(\varphi(e_1, \cdot))} + \dim U$

$\varphi(e_1, e_1) \neq 0$

$$\Longrightarrow \boxed{\dim U = n - 1}$$

$$\cdot\ U + \langle e_1 \rangle = U \oplus \langle e_1 \rangle$$

Indeed, $v \in \langle e_1 \rangle \cap U$

$$\Rightarrow \left| \begin{array}{l} v = \lambda e_1 \\ \lambda \in \mathbb{F} \end{array} \right. , \quad \varphi(e_1, v) = 0$$

$$\varphi(e_1, v) = \varphi(e_1, \lambda e_1) = \lambda \underbrace{\varphi(e_1, e_1)}_{\neq 0}$$

$$\Rightarrow \lambda = 0 \Rightarrow v = 0 .$$

$$U + \langle e_1 \rangle = \underset{\substack{\uparrow \\ \dim U = n-1}}{U} \oplus \underset{\dim 1}{\underline{\langle e_1 \rangle}}$$

$$\Rightarrow \boxed{V = \langle e_1 \rangle \oplus U} \Leftarrow \text{FUNDAMENTAL}$$

. Pick $(e_2, \dots, e_n)$ basis of U

$\mid \mathcal{B} = (e_1, \dots, e_n)$ basis of V

and :

$$[\varphi]_B = \begin{array}{c} \left(\begin{array}{c|ccc} \varphi(e_1,e_1) & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array}\right) \end{array} \begin{array}{c} e_1 \\ \vdots \\ \vdots \\ e_n \end{array}$$

$\varphi(e_1, e_j) = 0$, $j \neq 1$

$\varphi(e_j', e_1) = 0$, $j \neq 1$

$$A' = (\varphi(e_i', e_j'))_{\substack{2 \le i \le n \\ 2 \le j \le n}} \qquad A' = {}^T A'$$

$A' = [\varphi]_{B'}$ $\quad B' = (e_2', \dots, e_n')$

$\varphi|_U : U \times U \to F$ is symmetric with matrix A'.

$\Rightarrow$ apply the induction hypothesis on U

$\Rightarrow$ find $(e_2', \dots, e_n')$ such that

$\varphi|(e_1', \dots, e_n')$ diagonal

$\Rightarrow$ φ diagonal in $(e_1, e_2', \dots, e_n')$ $\square$

Example $V = \mathbb{R}^3$, e_1, e_2, e_3.

. $Q(x_1, x_2, x_3) = 1x_1^2 + 1x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$

$= X^T A X$, $\boxed{A = A^T}$.

. $$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

. **Diagonalize it**

① Follow the proof of diagonalization

$\rightsquigarrow$ algorithm

② "Complete the square":

$$Q(x_1, x_2, x_3) = \cancel{x_1^2} + \cancel{x_2^2} + 2x_3^2 + \cancel{2x_1x_2} + \cancel{2x_1x_3} - 2x_2x_3$$

$$= (x_1 + x_2 + x_3)^2 + x_3^2 - 4x_2x_3$$

$$= \underbrace{(x_1 + x_2 + x_3)}_{x_1'}{}^2 + \underbrace{(x_3 - 2x_2)}_{x_2'}{}^2 - \underbrace{(2x_2)}_{x_3'}{}^2$$

$$\Rightarrow P, \quad P^TAP = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -1 \end{pmatrix} \to -4$$

To find P, notice that:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \overbrace{\boxed{\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & 0 \end{pmatrix}}}^{P^{-1}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

D.