Lecture 16 Cayley – Hamilton Theorem and multiplicity of eigenvalues

. Minimal polynomial $\alpha \in L(V)$, $p \in F[t]$

$p(\alpha) = 0 \iff m_\alpha \mid p$ (m_α divides p)

. $m_\alpha \equiv$ polynomial with smallest degree which kills α is well defined (normalized by assuming that the coefficient of highest degree is 1).

Th (Cayley – Hamilton)

Let V F vspace, $\dim_F V < +\infty$. Let

$\alpha \in L(V)$ with characteristic polynomial:

$$\chi_\alpha(t) = \det(\alpha - t\,\mathrm{Id})$$

Then

$$\chi_\alpha(\alpha) = 0$$

Cor $m_\alpha \mid \chi_\alpha$ (m_α divides χ_α)

proof We give two proofs.

① $\mathbb{F} = \mathbb{C}$ $\quad B = \{v_1, \dots, v_n\}$

$n = \dim_{\mathbb{C}} V$

$$[\alpha]_B = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \quad (\text{triangulable})$$

Let: $U_j = \langle v_1, \dots, v_j \rangle$. Then:

$$(\alpha - a_j \mathrm{Id})\, U_j \leq U_{j-1}$$

$$\begin{array}{cc} \begin{matrix} v_1 & \cdots & v_j \end{matrix} & \\ \begin{pmatrix} a_1 & & & * \\ & \ddots & & \\ & & a_j & \\ 0 & & & \ddots \\ & & & & a_n \end{pmatrix} & \left.\begin{matrix} v_1 \\ \vdots \\ v_j \end{matrix}\right\} U_j \end{array}$$

$$\chi_\alpha(t) = \prod_{i=1}^{n} (a_i - t)$$

$$(\alpha - a_1 \mathrm{Id}) \underbrace{\cdots \underbrace{(\alpha - a_{n-1}\mathrm{Id}) \underbrace{(\alpha - a_n \mathrm{Id}) V}_{\leq U_{n-1}}}_{\leq U_{n-2}}}_{\cdots}$$

$$\leq (\alpha - a_1 \mathrm{Id})\, U_1 = 0.$$

$$\Rightarrow \chi_\alpha(\alpha) = 0.$$

② Any field F.

. $A \in M_n(F)$, $(-1)^n \chi_A(t) = \det(t\,Id - A)$

$$= t^n + a_{n-1}t^{n-1} + \dots + a_1 t + a_0 \leftarrow \det B$$

. For any matrix B: $\left(B \cdot adj(B) = (\det B)\, Id \right)$ $(*)$

. We apply to: $B = t\,Id - A$.

$adj(t\,Id - A) =$ matrix with entries polynomials of degree $< n$.

$(*)$ $(t\,Id - A)\left[B_{n-1}t^{n-1} + \dots + B_1 t + B_0\right]$

$$= \underbrace{\left(t^n + a_{n-1}t^{n-1} + \dots + a_1 t + a_0\right)}_{\det B} Id$$

<u>Equate coeff</u>

t^n $\left(Id = B_{n-1} \right)$

t^{n-1} $\left(a_{n-1} I = B_{n-2} - AB_{n-1} \right)$

$\vdots$

$$t^0 \left(a_0 I = -AB_0 \right)$$

I can let $t = A$ in these relations:

$$\boxed{\begin{aligned} A^n \times & \left(I = B_{n-1} \right) \\ A^{n-1} \times & \left(a_{n-1} I = B_{n-2} - AB_{n-1} \right) \\ & \vdots \\ A^0 \times & \left(a_0 I = -AB_0 \right) \end{aligned}}$$

sum

$$\Rightarrow \underbrace{A^n + a_{n-1} A^{n-1} + \dots + a_0 I}_{} = 0$$

$$\chi_A(A) = 0.$$

$\square$

Def (algebraic / geometric multiplicity)

$\alpha \in L(V)$, λ eigenvalue of α.

. Then:

$$\chi_\alpha(t) = (t-\lambda)^{a_\lambda} q(t)$$

$$q \in F[t], \quad (t-\lambda) \nmid q \quad (q(\lambda) \neq 0)$$

$a_\lambda \equiv$ algebraic multiplicity of λ

. $g_\lambda \equiv$ geometric multiplicity of λ

$= \dim \operatorname{Ker}(\alpha - \lambda \mathrm{Id})$

Rk λ eigenvalue $\iff$ $\alpha - \lambda \mathrm{Id}$ singular

$\iff \chi_\alpha(\lambda) = \det(\alpha - \lambda \mathrm{Id}) = 0.$

Lemma λ eigenvalue of $\alpha \in L(V)$,

$$1 \leq g_\lambda \leq a_\lambda .$$

proof • $g_\lambda = \dim \operatorname{Ker}(\alpha - \lambda \mathrm{Id}) \geq 1$;
λ eigenvalue.

• Let us show that $g_\lambda \leq a_\lambda$. Indeed, let $(v_1, \dots, v_{g_\lambda})$ be a basis of $V_\lambda = \operatorname{Ker}(\alpha - \lambda \mathrm{Id})$, and complete it to:

$$\mathcal{B} = (v_1, \dots, v_{g_\lambda}, v_{g_\lambda + 1}, \dots, v_n) \text{ of } V$$

Then

$$[\alpha]_{\mathcal{B}} = \left(\begin{array}{c|c} \lambda \mathrm{Id}_{g_\lambda} & * \\ \hline 0 & A_1 \end{array}\right) \quad \text{for some } A_1 .$$

$$\Rightarrow \det(\alpha - t\,\mathrm{Id}) = \det\left(\begin{array}{c|c} (\lambda - t)\,\mathrm{Id}_{g_\lambda} & * \\ \hline 0 & A_1 - t\,\mathrm{Id} \end{array}\right)$$

$$= (\lambda - t)^{g_\lambda} \underbrace{\chi_{A_1}(t)}_{\text{polynomial}} \Rightarrow \boxed{a_\lambda \geq g_\lambda}$$

determinant by the

D.

Lemma Let λ be an eigenvalue of α. Let $c_\lambda \equiv$ multiplicity of λ as a root of the minimal polynomial m_α.

Then $1 \leq c_\lambda \leq a_\lambda$.

proof. Cayley-Hamilton: $m_\alpha \mid \chi_\alpha$

$$\Rightarrow c_\lambda \leq a_\lambda$$

- $\underline{c_\lambda \geq 1}$ Indeed, λ eigenvalue,

$\exists v \neq 0 \ / \ \alpha(v) = \lambda v$.

For such an eigenvector, $\alpha^p(v) = \lambda^p v$

$\Rightarrow \forall p \in F[t], \quad p(\alpha)(v) = [p(\lambda)] v$

I take $p = m_\alpha$

$$\Rightarrow \underbrace{m_\alpha(\alpha)}_{\substack{\parallel \\ 0}}(v) \underset{\substack{\uparrow \\ v \text{ eigenvector}}}{=} [m_\alpha(\lambda)] \underset{\neq 0}{v}$$

$\Rightarrow m_\alpha(\lambda) = 0$

$\Rightarrow t - \lambda \mid m_\alpha \quad \Rightarrow \quad c_\lambda \geq 1$. $\qquad$ D.

<u>Ex 1</u>

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

<u>m_A ?</u>

- $\chi_A(t) = (t-1)^2(t-2)$
  (Compute the determinant)

- m_A is either: a) $(t-1)^2(t-2)$
  b) $(t-1)(t-2)$

<u>Check</u> $(A-I)(A-2I) = 0$

$\Longrightarrow$ b) holds, $m_A = (t-1)(t-2)$

$\Longrightarrow$ A diagonalizable.

<u>Ex 2</u>

$$A = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

"Jordan block"

$\in M_n(F)$

<u>Check</u> $g_\lambda = 1$, $a_\lambda = n$, $c_\lambda = n$.

Ex3 $A = \lambda I_d$ $\qquad g_\lambda = n,\ a_\lambda = n,\ c_\lambda = 1.$

Lemma (Characterization of diagonalizable endomorphisms on $F = \mathbb{C}$)

V F vector space, $\dim V = n < +\infty$
$\alpha \in L(V)$

$\boxed{F = \mathbb{C}}$

TFAE:

(i) α is diagonalizable
(ii) $\forall \lambda$ eigenvalue of α, $a_\lambda = g_\lambda$
(iii) $\forall \lambda$ eigenvalue of α, $c_\lambda = 1$

proof (i) $\Leftrightarrow$ (iii) We have already done it
(i) $\Leftrightarrow$ (ii)

Indeed, let $(\lambda_1, \dots, \lambda_k)$ be the distinct eigenvalues of α. We showed:

always true

$$\alpha \text{ diagonalizable} \iff V = \underbrace{\bigoplus_{i=1}^{k} V_{\lambda_i}}$$

$$\dim V = n = \deg \chi_\alpha \underset{\substack{\uparrow \\ F=\mathbb{C}}}{=} \sum_{i=1}^{k} a_{\lambda_i}$$

$$\dim\left(\bigoplus_{i=1}^{k} V_{\lambda_i}\right) = \sum_{i=1}^{k} g_{\lambda_i}$$

$$\left(\chi_\alpha(t) = (-1)^n \prod_{i=1}^{k} (t-\lambda_i)^{a_{\lambda_i}}\right)$$

α diagonalizable

$$\iff \sum_{i=1}^{k} a_{\lambda_i} = \sum_{i=1}^{k} g_{\lambda_i} \qquad (*)$$

We know that: $\forall\, 1 \leq i \leq k,\ g_{\lambda_i} \leq a_{\lambda_i}$

Hence $(*)$ holds

$\iff \forall\, 1 \le i \le k,\quad a_{\lambda_i} = g_{\lambda_i}$.

D.

Summary over $\boxed{\mathbb{C} = F}$.

$$\chi_\alpha(t) = (t-\lambda_1)^{a_{\lambda_1}} \cdots (t-\lambda_k)^{a_{\lambda_k}}$$

$$m_\alpha(t) = (t-\lambda_1)^{c_{\lambda_1}} \cdots (t-\lambda_k)^{c_{\lambda_k}} \qquad 1 \le c_{\lambda_i} \le a_{\lambda_i}$$

$$g_{\lambda_i} = \dim \ker(\alpha - \lambda_i \mathrm{Id})$$