Lecture 15

Diagonalization criterion and minimal polynomial

Notation

$p(t)$ polynomial over F

$p(t) = a_n t^n + \dots + a_1 t + a_0, \quad a_i \in F$

- $A \in M_n(F)$, we define:

$$p(A) = a_n A^n + \dots + a_1 A + a_0 \mathrm{Id} \in M_n(F)$$

- $\alpha \in L(V)$, we define:

$$p(\alpha) = a_n \alpha^n + \dots + a_1 \alpha + a_0 \mathrm{Id}$$

$$\alpha^{\delta} = \underbrace{\alpha \circ \dots \circ \alpha}_{\delta} \in L(V),$$

Thm (Sharp criterion of diagonalizability)

- V vector space over F, $\dim V < +\infty$
- $\alpha \in L(V)$

Then α is diagonalizable
$\iff$ $\exists$ a polynomial p which is the product of distinct linear factors such that $p(\alpha) = 0$.

α diag $\iff$ $\exists (\lambda_1, \dots, \lambda_k)$ distinct /
$p(t) = \prod_{i=1}^{k} (t - \lambda_i)$
$p(\alpha) = 0$.

proof $\implies$) Suppose that α is diagonalizable, with distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Let

$$p(t) = \prod_{i=1}^{k} (t - \lambda_i)$$

Let $v \in B$
B = basis of V formed of eigenvectors.

Then $v \in \underline{B}$ $\alpha(v) = \lambda_i v$ for some i

$$\Rightarrow \quad (\alpha - \lambda_i \mathrm{Id}) v = 0$$

$$\Rightarrow \quad p(\alpha) = \left[\prod_{i=1}^{k} (\alpha - \lambda_i \mathrm{Id})\right] v = 0 .$$

these terms commute

$$(\alpha - \lambda_i \mathrm{Id})(\alpha - \lambda_j \mathrm{Id}) = (\alpha - \lambda_j \mathrm{Id})(\alpha - \lambda_i \mathrm{Id})$$

$$\Rightarrow \forall v \in \underline{B} \quad p(\alpha)(v) = 0$$

$$\Rightarrow \quad p(\alpha) = 0 .$$

$\Leftarrow$) Suppose $p(\alpha) = 0$ for some:

$$p(t) = \prod_{i=1}^{k} (t - \lambda_i)$$

$$\lambda_i \neq \lambda_j, \quad i \neq j .$$

Let $V_{\lambda_i} = \ker(\alpha - \lambda_i \mathrm{Id})$, We claim $\boxed{V = \bigoplus_{i=1}^{k} V_{\lambda_i} \quad (*)}$

Indeed let: $q_j(t) = \prod_{\substack{i=1 \\ i \neq j}}^{k} \left(\dfrac{t - \lambda_i}{\lambda_j - \lambda_i} \right)$, $1 \leq j \leq k$

then: $q_j(\lambda_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

Hence the polynomial $q(t) = \sum_{j=1}^{k} q_j(t)$ has degree $\leq k-1$ and:

$$q(\lambda_j) = 1 \quad , \quad 1 \leq j \leq k$$

$\Rightarrow \forall t, \quad q(t) = 1 \Rightarrow q_1(t) + \dots + q_k(t) = 1$.

Let: $\pi_j = q_j(\alpha) \in L(V)$, then:

$$\sum_{j=1}^{k} \pi_j = \left(\sum_{j=1}^{k} q_j \right)(\alpha) = \mathrm{Id}.$$

This means: $\forall v \in V$,

$$v = q(\alpha)(v) = \sum_{j=1}^{k} \pi_j(v) = \sum_{j=1}^{k} q_j(\alpha)(v)$$

$(*)$

<u>Observe</u>

$$(\alpha - \lambda_j \mathrm{Id})\, q_j(\alpha)(v)$$

$$= \frac{1}{\prod_{i \neq j} (\lambda_j - \lambda_i)} \, p(\alpha)(v) = 0$$

$\Rightarrow \forall j \in \{1, \dots, k\}, \quad \pi_j(v) \in V_{\lambda_j}$

$$\underset{(*)}{\Rightarrow} \quad V = \sum_{j=1}^{k} V_{\lambda_j} .$$

. It remains to prove that the sum is direct.

Indeed, let: $v \in V_{\lambda_j} \cap \left(\sum_{i \neq j} V_{\lambda_i} \right)$

Let's apply π_j to $v \in \boxed{V_{\lambda_j} \cap \left(\sum_{i \neq j} V_{\lambda_i}\right)}$

. $v \in V_{\lambda_j} \Rightarrow \pi_j(v) = \left(\prod_{i \neq j} \left(\frac{\lambda_j - \lambda_i}{\lambda_j - \lambda_i}\right)\right) = v$

. $v \in \sum_{i \neq j} V_{\lambda_i}$, $\Big|$ $w \in V_{\lambda_i}$, $i \neq j$

$\Rightarrow \pi_j(w) = 0$.

$\Rightarrow \pi_j(v) = 0$.

This implies: $v = \pi_j(v) = 0$

$\Rightarrow$ the sum is direct.

$\Rightarrow v$ is diagonalizable. $\square$.

Remark We have shown the following: if $\lambda_1, \ldots, \lambda_k$ are k distinct eigenvalues of α,

then the sum: $\boxed{\sum_{i=1}^{k} V_{\lambda_i} = \bigoplus_{i=1}^{k} V_{\lambda_i}}$

always true. The only way diagonalization fails is if $\sum_{j=1}^{k} V_{\lambda_j} \lneq V$.

Ex If $A \in M_n(F)$ has finite order $\left(\equiv \left(A^m = Id \text{ for some } m \in \mathbb{N} \right) \right)$

Then A is diagonalizable.

proof $t^m - 1 = \prod_{j=0}^{m-1} (t - \zeta_m^j)$

$\zeta_m = e^{\frac{2i\pi}{m}}$.

Thm (Simultaneous diagonalization)

Let $\alpha, \beta \in L(V)$ diagonalizable. Then α, β are simultaneously diagonalizable (ie there exist a basis in which both matrices are diagonal) iff α and β commute.

proof $\Longrightarrow$) $\exists\, \mathcal{B}$, $\begin{cases} [\alpha]_{\mathcal{B}} = D_1 \\ [\beta]_{\mathcal{B}} = D_2 \end{cases}$

D_1, D_2 diagonal, then $D_1 D_2 = D_2 D_1$

$\Rightarrow \quad \alpha\beta = \beta\alpha$

$\Longleftarrow$) Suppose α, β diagonalizable and commute.

$\begin{cases} V = V_{\lambda_1} \oplus \dots \oplus V_{\lambda_e} \\ \lambda_1, \dots, \lambda_e = \text{the distinct eigenvalues of } \alpha \end{cases}$

claim $\boxed{\beta(V_{\lambda_j}) \leq V_{\lambda_j}}$.

Indeed, let $v \in V_{\lambda_j}$. Then:

$$\alpha\beta(v) = \beta\,\alpha(v) = \beta(\lambda_j v) = \lambda_j\,\beta(v)$$

$\Rightarrow \beta(v) \in V_{\lambda_j}$.

Since β is diagonalizable, $\exists\, p$ w/ distinct linear factors such that $p(\beta) = 0$.

<u>Now</u> $p(\beta_{|V_{\lambda_i}}) = p(\beta_{|V_{\lambda_i}}) = 0$

$$\Rightarrow \boxed{\beta_{|V_{\lambda_i}}} \in L(V_{\lambda_i}) \text{ diagonalizable.}$$

I take $\mathcal{B}_i$ basis of V_{λ_i} in which $\beta_{|V_{\lambda_i}}$ is diagonalizable. Since the sum is direct, $(\underset{V_{\lambda_1}}{\underset{\downarrow}{\mathcal{B}_1}}, \dots, \underset{V_{\lambda_k}}{\underset{\downarrow}{\mathcal{B}_k}}) \equiv$ basis of V

in which β is diagonal, and so is α. $\square$

Minimal polynomial

Remainder (Group - Ring - Modules)

- Euclidian algorithm for polynomials: given a, b polynomials over F with $b \neq 0$, there exist polynomials q, r over F with:

$$\deg r < \deg b \quad \text{and:}$$

$$a = qb + r$$

remainder.

Def (Minimal polynomial)

V F vector space, $\alpha \in L(V)$, $\dim V < +\infty$.
The minimal polynomial m_α of α is the non zero polynomial with smallest degree

$$\boxed{\text{such that: } m_\alpha(\alpha) = 0.}$$

Rm $\dim_F V = n < +\infty$, $\alpha \in L(V)$.

$\dim_F L(V) = n^2$, hence:

$\underbrace{Id, \alpha, \alpha^2, \dots, \alpha^{n^2}}_{n^2+1 \text{ terms}}$ are linearly dependent, so:

$a_{n^2} \alpha^{n^2} + \dots + a_1 \alpha + a_0 = 0$

$(a_{n^2}, \dots, a_0) \neq (0, \dots, 0)$

Lemma $\alpha \in L(V)$, $p \in F[t]$.

Then: $p(\alpha) = 0$ iff m_α is a factor of p.

(in particular, m_α is well defined) proof

$p \in F[t] \;/\; p(\alpha) = 0$

m_α, $m_\alpha(\alpha) = 0$, $\deg m_\alpha \leq \deg p$

By Euclidean division:

$$p = m_\alpha q + r$$

$$\boxed{\deg r < \deg m_\alpha}$$

Then: $p(\alpha) = 0 = \underbrace{m_\alpha(\alpha)}_{= 0} q(\alpha) + r(\alpha)$

$\Rightarrow r(\alpha) = 0 \Rightarrow r \equiv 0$

minimality of m_α in terms of degree

. If m_1, m_2 both minimal, then

m_1 divides m_2 and m_2 divides m_1

$\Rightarrow m_1 = c\, m_2$, $c \in F$.

Ex $V = F^2$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

. $p(t) = (t-1)^2$, $p(A) = p(B) = 0$

. min polynomial is either $(t-1)^2$ or $(t-1)$.

. <u>clear</u> $m_A = t-1$

$m_B = (t-1)^2$

$\Rightarrow$ A diagonalizable

B is <u>not</u> diagonalizable.