Lecture 14 Eigenvectors, eigenvalues and trigonal matrices

→ first step towards the diagonalization of endomorphisms.

- V vector space over F, $\dim V = n < +\infty$.
  $\alpha : V \to V$ linear $\equiv$ endomorphism of V.

- General problem: Can we find a basis $\mathcal{B}$ of V such that in this basis,

$$[\alpha]_{\mathcal{B}} \underset{\text{def}}{\equiv} [\alpha]_{\mathcal{B},\mathcal{B}}$$

has a "nice" form.

Remainder $\mathcal{B}'$ another basis

P : change of basis matrix

$$[\alpha]_{\mathcal{B}'} = P^{-1} [\alpha]_{\mathcal{B}} P .$$

Equivalently, given a square matrix $A \in M_n(F)$ is it conjugated to a matrix with a "simple" form.

Def (i) $\alpha \in L(V)$ ($\alpha : V \to V$ linear) is diagonalizable if there exist a basis $\mathcal{B}$ of V such that: $[\alpha]_{\mathcal{B}}$ diagonal

$$\left(\Leftrightarrow [\alpha]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \right)$$

(ii) $\alpha \in L(V)$ is triangulable if there exists $\mathcal{B}$ basis of V such that $[\alpha]_{\mathcal{B}}$ is triangular

$$\left([\alpha]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \right)$$

Rk A matrix is diagonalizable (triangulable) iff it is conjugate to a diagonal (triangular) matrix.

Def (i) $\lambda \in F$ is an eigenvalue of $\alpha \in L(V)$ iff :

$\exists\, v \in V \setminus \{0\}$ s.t. $\alpha(v) = \lambda v$.

(ii) $v \in V$ is an eigenvector of α iff :

$v \neq 0$ and : $\exists \lambda \in F \;/\; \alpha(v) = \lambda v$.

(iii) $V_\lambda = \{ v \in V \;/\; \alpha(v) = \lambda v \} \leq V$

is the eigenspace associated to λ .

Rk I will use the short notation evector, evalue, espace.

Lemma $\alpha \in L(V)$, $\lambda \in F$

λ evalue $\iff \det(\alpha - \lambda \mathrm{Id}) = 0$.

proof λ evalue $\iff \exists v \in V \setminus \{0\} \;/$

$\alpha(v) = \lambda v \qquad \left((\alpha - \lambda \mathrm{Id})(v) = 0 \right)$

$\iff \ker(\alpha - \lambda \mathrm{Id}) \neq \{0\}$

$\iff \alpha - \lambda \mathrm{Id}$ not injective

$\iff \alpha - \lambda \mathrm{Id}$ not surjective

$\uparrow$

endomorphism

$\iff$ ——— not bijective

$\iff \det(\alpha - \lambda \mathrm{Id}) = 0.$ $\square$

Rk (i) If $\left| \begin{array}{l} \alpha(v_j) = \lambda v_j, \\ v_j \neq 0 \end{array} \right.$ then

Complete into a basis $(v_1, \ldots, v_j, \ldots, v_n) = B,$

$$[\alpha]_B = \begin{array}{c} \begin{matrix} & j & \end{matrix} \\ \begin{pmatrix} \Big| & 0 & \Big| \\ \Big| & \vdots & \Big| \\ \Big| & \lambda & \Big| \\ \Big| & \vdots & \Big| \\ \Big| & 0 & \Big| \end{pmatrix} \end{array} \begin{matrix} \\ \\ \leftarrow j \\ \\ \\ \end{matrix}$$

Elementary facts about polynomials $\quad F$ field

- $f(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$
  $a_i \in F$

n largest exponent s.t. $a_n \neq 0$
  $n = \deg f$

- $\deg(f+g) \leq \max\{\deg f, \deg g\}$
  $\deg(fg) = \deg f + \deg g$

- $F[t] = \{$ polynomials with coeff. in $F \}$
- λ root of $f(t) \iff f(\lambda) = 0$.

$$\boxed{\begin{array}{l}\textbf{Lemma} \quad \lambda \text{ root of } f, \text{ then: } (t-\lambda) \text{ divides } f(t): \\ f(t) = (t-\lambda)\, g(t), \quad g \in F[t].\end{array}}$$

proof $f(t) = a_n t^n + \dots + a_1 t + a_0$

$$f(\lambda) = a_n \lambda^n + \dots + a_1 \lambda + a_0 = 0$$

$$\Rightarrow f(t) = f(t) - f(\lambda)$$

$$= a_n \underbrace{(t^n - \lambda^n)}_{(t-\lambda)(t^{n-1} + \lambda t^{n-2} + \dots + \lambda^{n-2} t + \lambda^{n-1})} + \dots + a_1 (t - \lambda)$$

$\square$.

Rk We say that λ is a root of **multiplicity k** if $(t-\lambda)^k$ divides f, but $(t-\lambda)^{k+1}$ does not.

Ex $f(t) = (t-1)^2 (t-2)^3$

Cor A non zero polynomial of degree n (≥ 0) has at most n roots (counted with multiplicity)

proof $\rightarrow$ exercise (induction on the degree) $\square$

Cor f_1, f_2 pol. of degree $< n$ s.t.:

$$\left| \begin{array}{l} f_1(t_i) = f_2(t_i) \\ (t_i)_{1 \leq i \leq n} \text{ are } n \text{ distinct values} \end{array} \right.$$

$$\Longrightarrow \quad f_1 \equiv f_2 .$$

proof $f_1 - f_2$ has degree $< n$ and at least n roots $\Rightarrow f_1 \equiv f_2$. $\square$

Th Any $f \in \mathbb{C}[t]$ of positive degree has a (complex) root.
(Hence exactly $\deg f$ roots when counted w. multiplicity)

$\to f \in \mathbb{C}[t]$,

$$f(t) = c \prod_{i=1}^{r} (t - \lambda_i)^{\alpha_i}, \quad c \in \mathbb{C}, \quad \lambda_i \in \mathbb{C}.$$

$\Longrightarrow$ Complex analysis.

Def $\alpha \in L(V)$, the characteristic polynomial of α is:

$$\chi_\alpha(t) = \det(A - \lambda I)$$

$$A - \lambda \mathrm{Id} = \begin{pmatrix} a_{11}-\lambda & & & \\ & a_{22}-\lambda & & a_{ij} \\ & & \ddots & \\ a_{ij} & & & a_{nn}-\lambda \end{pmatrix}$$

$\det(A - \lambda \mathrm{Id})$ is a polynomial in λ follows from the very definition of det.

Rk Conjugate matrices have the same characteristic polynomial:

$$\det(P^{-1}AP - \lambda \mathrm{Id})$$
$$= \det\left[P^{-1}(A - \lambda \mathrm{Id})P\right) = \det(A - \lambda \mathrm{Id}) \,\square.$$

Thm $\alpha \in L(V)$ is triangulable iff $\chi_\alpha(t)$ can be written as a product of linear factors over F:

$$\chi_\alpha(t) = c \prod_{i=1}^{n} (t - \lambda_i)$$

$\leadsto$ If $F = \mathbb{C}$, every matrix is triangulable

proof $\boxed{\Rightarrow}$ Suppose α triangulable,

$$[\alpha]_B = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

$$\Rightarrow \chi_\alpha(t) = \det \begin{pmatrix} a_1 - t & & * \\ & \ddots & \\ 0 & & a_n - t \end{pmatrix}$$

$$= \prod_{i=1}^{n} (a_i - t) \qquad \circ .$$

$\boxed{\Leftarrow}$ We argue by induction on $n = \dim V$.

- $n = 1$ ✓

- $n > 1$. By assumption, let $\chi_\alpha(t)$ have a root λ. (

$$\chi_\alpha(\lambda) = 0 \iff \lambda \text{ evalue of } \alpha \quad)$$

Let $U = V_\lambda$ = espace.

Let $(v_1, \dots, v_k)$ be a basis of U

We complete to $(v_{k+1}, \dots, v_n)$ basis of V.

$$\text{Span}(v_{k+1}, \dots, v_n) = W$$

$$V = U \oplus W$$

$$[\alpha]_B = \begin{array}{c|c} \begin{array}{cccccc} v_1 \cdots & v_k & v_{k+1} & \cdots & v_n \end{array} & \\ \left(\begin{array}{c|c} \lambda \mathrm{Id} & * \\ \hline 0 & C \end{array}\right) & \begin{array}{c} v_1 \\ \vdots \\ v_k \\ v_{k+1} \\ \vdots \\ v_n \end{array} \end{array}$$

α induces an endomorphism:

$$\bar{\alpha} : V/U \longrightarrow V/U$$

$$C = [\bar{\alpha}]_{\bar{B}}, \quad \bar{B} = (v_{k+1}+U, \dots, v_n + U)$$

Then (block product):

$$\det(\alpha - t\,\mathrm{Id}) = \det \left(\begin{array}{c|c} (\lambda - t)\mathrm{Id} & * \\ \hline 0 & C - t\,\mathrm{Id} \end{array}\right)$$

$$= (\lambda - t)^k \det(C - t\,\mathrm{Id})$$

(H) $= c \prod_{i=1}^{n} (t - a_i) \implies \det(C - t\,\mathrm{Id}) = \tilde{c} \prod_{i=k+1}^{n} (t - \tilde{a}_i)$

(because $\det(C - t\,\mathrm{Id})$ is a polynomial in t).

Induction $\left(\dim(V \backslash U) = n - \ell < n \right)$

$$\exists \underbrace{(\check{v}_{\ell+1}, \dots, \check{v}_n)}_{\substack{\shortparallel \\ \check{\mathcal{B}} \text{ basis of } W}} \Big/ \quad [\bar{\alpha}]_{\check{\mathcal{B}}} = \begin{pmatrix} \check{a}_{\ell+1} & & * \\ & \ddots & \\ 0 & & \check{a}_n \end{pmatrix}$$

and then: $\left| \begin{array}{l} V = U \oplus W \\ W = \mathrm{Span}(\check{v}_{\ell+1}, \dots, \check{v}_n) \end{array} \right.$

We may then compute the matrix of α in the basis

$$\hat{\mathcal{B}} = (v_1, \dots, v_\ell, \check{v}_{\ell+1}, \dots, \check{v}_n)$$

which has the form:

Then: $[\alpha]_{\tilde{B}} = $

$$\begin{array}{c c}
 & \begin{array}{cccccc} v_1 & \cdots & v_k & \tilde{v}_{k+1} & \cdots & \tilde{v}_n \end{array} \\
\begin{array}{c} v_1 \\ \vdots \\ v_k \\ \tilde{v}_{k+1} \\ \vdots \\ \tilde{v}_n \end{array} & \left(\begin{array}{c|c} \lambda I_k & * \\ \hline 0 & \begin{matrix} \ddots & * \\ 0 & \ddots \end{matrix} \end{array}\right)
\end{array}$$

$\Rightarrow$ triangular form. □

Lemma V n dimensional over $F = \mathbb{R}, \mathbb{C}$

$\alpha \in L(V)$

Say $\chi_\alpha(t) = (-1)^n t^n + \boxed{c_{n-1}} t^{n-1} + \dots + \boxed{c_0}$

Then:

$c_0 = \det A$

$c_{n-1} = (-1)^{n-1} \operatorname{tr} A$

proof $\chi_\alpha(t) = \det(A - t\,\mathrm{Id})$

$$\Rightarrow \chi_\alpha(0) = \det A.$$

Say that $F = \mathbb{R}$ or $\mathbb{C}$. (If $F = \mathbb{R}$, we can think of the matrix of having complex entries as well). We know that α is triangulable over $\mathbb{C}$:

$$\chi_\alpha(t) = \det \begin{pmatrix} a_1 - t & & * \\ & \ddots & \\ 0 & & a_n - t \end{pmatrix}$$

$$= \prod_{i=1}^{n} (a_i - t) = (-1)^n t^n + c_{n-1} t^{n-1} + \dots + c_0$$

$$c_{n-1} = (-1)^{n-1} \underbrace{\sum_{i=1}^{n} a_i}_{\mathrm{Tr}\,\alpha}$$

D.

proof $\chi_\alpha(t) = \det(A - t\,\mathrm{Id})$

$\Rightarrow \chi_\alpha(0) = \det A$.

Say that $F = \mathbb{R}$ or $\mathbb{C}$. (If $F = \mathbb{R}$, we can think of the matrix of having complex entries as well). We know that α is triangulable over $\mathbb{C}$:

$$\chi_\alpha(t) = \det \begin{pmatrix} a_1 - t & & * \\ & \ddots & \\ 0 & & a_n - t \end{pmatrix}$$

$$= \prod_{i=1}^{n} (a_i - t) = (-1)^n t^n + c_{n-1} t^{n-1} + \dots + c_0$$

$$c_{n-1} = (-1)^{n-1} \underbrace{\sum_{i=1}^{n} a_i}_{\mathrm{Tr}\,\alpha}$$

□.