Lecture 13 Adjugate matrix

- **Observation** $A \in M_n(F)$

$$A = \left(A^{(1)} | \dots | A^{(n)}\right)$$

- Action of the determinant when swapping two column vectors: $1 \le j < k \le n$

$$\det\left(A^{(1)} | \dots | A^{(j)} | \dots | A^{(k)} | \dots | A^{(n)}\right)$$

$$= - \det\left(A^{(1)} | \dots | A^{(k)} | \dots | A^{(j)} | \dots | A^{(n)}\right)$$

alternate n linear form

- Using that $\det A = \det(A^T)$, we can see that swapping two lines changes the determinant by a factor (-1).

<u>Remark</u> We could prove properties of determinant using the decomposition of A into elementary matrices.

Column (line) expansion and adjugate matrix

Column expansion $\equiv$ to reduce the computation of $n \times n$ determinants to $(n-1) \times (n-1)$ determinants.

$\rightsquigarrow$ very useful to compute determinants

Def $A \in M_n(F)$

Pick i, j, $\left| \begin{array}{l} i \in \{1, \dots, n\} \\ j \in \{1, \dots, n\} \end{array} \right.$

We define: $A_{\hat{i}\hat{j}} \in M_{n-1}(F)$ obtained by removing the i-th row and the j-th column from A.

Ex

$$A = \begin{array}{cc} & \begin{array}{ccc} & j=2 & \end{array} \\ & \begin{pmatrix} 1 & 2 & -7 \\ 2 & 1 & 0 \\ -3 & 6 & 1 \end{pmatrix} \begin{array}{c} \\ \\ i=3 \end{array} \end{array}$$

$$A_{\hat{3}\hat{2}} = \begin{pmatrix} 1 & -7 \\ 2 & 0 \end{pmatrix}$$

Lemma (Expansion of the determinant)

Let $A \in M_n(F)$.

(i) Expansion with respect to the j-th column:

pick $1 \leq j \leq n$, then:

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{\hat{ij}} \qquad (*)$$

(ii) Expansion with respect to the i-th row:

pick $1 \leq i \leq n$, then:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} \det A_{\hat{ij}}$$

$\Rightarrow$ powerful tool to compute determinants

Example

$$A = \begin{array}{ccc} + & - & + \\ \left(\begin{array}{ccc} 1 & 2 & -1 \\ 3 & -1 & 1 \\ 4 & 2 & -7 \end{array}\right) \end{array}$$

$$\begin{array}{ccc} - & + & - \\ + & - & + \end{array}$$

2nd column

$$\det A = -(2)\begin{vmatrix} 3 & 1 \\ 4 & -7 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -1 \\ 4 & -7 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}$$

proof Expansion with respect to the j-th column (row expansion formula follows by taking ^{T}).

Pick $1 \le j \le n$.

- $A = \left(A^{(1)} \middle| A^{(2)} \cdots \middle| A^{(j)} \middle| \cdots \middle| A^{(n)} \right)$

$$A^{(j)} = \sum_{i=1}^{n} a_{ij} e_i , \qquad A = (a_{ij})_{1 \le i,j \le n}$$

- $\det A =$

$$\det\left(A^{(1)}, \ldots, \sum_{i=1}^{n} a_{ij} e_i, \ldots, A^{(n)} \right)$$

$$= \sum_{i=1}^{n} a_{ij} \underbrace{\det\left(A^{(1)}, \ldots, \overset{j\text{th}}{e_i}, \ldots, A^{(n)} \right)}$$

$$\det \begin{array}{c} \\ \left(\begin{array}{c|c|c|c|c} & & j & & \\ A^{(1)} & \cdots & \begin{matrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & \cdots & A^{(n)} \end{array} \right) \end{array} \begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \\ \end{array}$$

$$= (-1)^{j-1} \det \begin{pmatrix} 0 & & & & & \\ \vdots & & & & & \\ 1 & A^{(1)} & \cdots & A^{(j-1)} & A^{(j+1)} & \cdots & A^{(n)} \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{pmatrix} \begin{array}{l} 1 \\ \\ i \end{array}$$

$$= (-1)^{i-1} (-1)^{j-1} \det \begin{pmatrix} \boxed{1} & a_{i1} & \cdots & a_{ij-1} & a_{ij+1} & \cdots & a_{in} \\ 0 & & & & & & \\ \vdots & & & \widehat{A_{ij}} \in \mathcal{M}_{n-1,n-1} & & & \\ 0 & & & & & & \end{pmatrix}$$

$$= (-1)^{i+j} \det\left(\widehat{A_{ij}}\right)$$

bloc determinant formula

We have proved:

$$\det\left(A^{(1)}, \dots, A^{(j-1)}, \overset{\overset{j}{\downarrow}}{e_i}, A^{(j+1)}, \dots, A^{(n)}\right)$$

$$= (-1)^{i+j} \det\left(A_{\hat{ij}}\right).$$

Now: $\det A = \sum_{i=1}^{n} a_{ij} \det\left(A^{(1)}, \dots, A^{(j-1)}, e_i, \dots, A^{(n)}\right)$

$$= \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det A_{\hat{ij}} \qquad \square$$

Def (Adjugate matrix)

Let $A \in M_n(F)$. The adjugate matrix $\operatorname{adj}(A)$ is the $n \times n$ matrix with (i,j) entry given by:

$$(-1)^{i+j} \det (A_{\hat{ji}}) \equiv \left[\det\left(A^{(1)}, \ldots, A^{(j-1)}, \underset{\substack{\| \\ (\operatorname{adj}(A))_{ji}}}{e_i}, A^{(j)}, \ldots, A^{(n)}\right)\right]$$

Thm Let $A \in M_n(F)$, then:

$$\operatorname{adj}(A)\, A = (\det A)\, I_d = \begin{pmatrix} \det A & & 0 \\ & \ddots & \\ 0 & & \det A \end{pmatrix}$$

In particular, when A is invertible,

$$\boxed{A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)},$$

proof We just proved: $(*)$

- $$\det A = \sum_{i=1}^{n} (-1)^{i+j} \left(\det A_{\hat{ij}}\right) a_{ij}$$

$$\underset{\substack{\uparrow \\ \text{def of} \\ \text{adj}(A)}}{=} \sum_{i=1}^{n} \left(\text{adj}(A)\right)_{ji} a_{ij} = \left(\text{adj}(A)\,A\right)_{jj}$$

- For $j \neq k$, we have:

$$0 = \det\big(A^{(1)}, \dots, \underset{\substack{\uparrow \\ j\text{-th Column} \\ \text{vector}}}{A^{(k)}}, \dots, A^{(k)}, \dots, A^{(n)}\big)$$

$$= \det\Big(A^{(1)}, \dots, \sum_{i=1}^{n} a_{ik} e_i, \dots, A^{(k)}, \dots, A^{(n)}\Big)$$

$$= \sum_{i=1}^{n} a_{ik} \underbrace{\left[\det \left(A^{(1)}, \dots, \underset{\substack{\uparrow \\ j}}{e_i}, \dots, A^{(n)} \right) \right]}_{= (\operatorname{adj}(A))_{ji}}$$

$$= \sum_{i=1}^{n} (\operatorname{adj}(A))_{ji} \, a_{ik}$$

$$= \left(\operatorname{adj}(A) \, A \right)_{jk} = 0 \quad \text{for } j \neq k.$$

Cramer rule

Prop Let $A \in M_n(F)$ be invertible. Let $b \in F^n$. Then the unique

Solution to $Ax=b$ is given by:

$$x_i = \frac{1}{\det A} \det\left(A_{\hat{i}} b\right), \quad 1 \leq i \leq n, \quad (*)$$

where: $A_{\hat{i}} b$ is obtained by replacing the i-th column of A by b.

$\rightsquigarrow$ Algorithmically, this avoids computing A^{-1}

proof A invertible, $\exists! \, x \in F^n \,/\, Ax = b$.
Let x be this solution, then:

$$\det\left(A_{\hat{i}} b\right) = \det\left(A^{(1)}, \dots, A^{(i-1)}, b, A^{(i+1)}, \dots, A^{(n)}\right)$$

$$= \det\left(A^{(1)}, \dots, A^{(i-1)}, Ax, A^{(i+1)}, \dots, A^{(n)}\right)$$

$$= \det\left(A^{(1)}, \dots, A^{(i-1)}, \sum_{j=1}^{n} x_j \cdot A^{(j)}, A^{(i+1)}, \dots, A^{(n)}\right)$$

alternate linear

$$= x_i \det\left(A^{(1)}, \dots, A^{(i-1)}, A^{(i)}, \dots, A^{(n)}\right)$$

$$= x_i \det A$$

$$\Rightarrow \quad x_i = \frac{\det \widehat{A_{ib}}}{\det A}, \qquad \square$$