Lecture 11 DETERMINANTS

Permutations and transpositions

. **permutation** : S_n = group of permutations of $\{1, \dots, n\}$

$\sigma : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ bijection

σ = permutation

. **transposition** : $k \neq \ell$, $\tau_{k\ell} \in S_n$

$$\tau_{k\ell} \downarrow \begin{pmatrix} 1 & \dots & k & \dots & \ell & \dots & n \\ \downarrow & & & & & & \downarrow \\ 1 & \dots & \ell & \dots & k & \dots & n \end{pmatrix}$$

Exchange k and ℓ.

. **decomposition** any permutation σ can be decomposed as a product of transpositions :

$$\sigma = \prod_{i=1}^{n_\sigma} \tau_i \quad , \quad \tau_i \text{ transposition}$$

. signature of a permutation:

$$\varepsilon : S_n \longrightarrow \{1, -1\}$$
$$\sigma \mapsto \begin{cases} 1 & \text{if } n_\sigma \text{ even} \\ -1 & \text{if } n_\sigma \text{ odd} \end{cases}$$

Then: ε is a homomorphism:

$$\varepsilon(\sigma \circ \sigma') = \varepsilon(\sigma)\,\varepsilon(\sigma')$$

Def $A \in M_n(F)$, $A = (a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}}$.

We define the determinant of A:

$$\det A = \sum_{\sigma \in S_n}' \varepsilon(\sigma) \underbrace{a_{\sigma(1)1}\, a_{\sigma(2)2} \cdots a_{\sigma(n)n}}$$

n! summands

one term for each row and column

Ex

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

- $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ $\quad \varepsilon = 1$

- $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad \varepsilon = -1$

Lemma If $A = (a_{ij})$ is an upper (lower) triangular matrix:

$$a_{ij} = 0 \text{ for } i \geq j \quad (\text{resp } i \leq j)$$

then: $\det A = 0$.

$$\det \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} = = \det \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix} = 0$$

proof $\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}$

For the summand not to be zero, we need:

$$\sigma(j) < j \quad \forall j \in \{1, \dots, n\}$$

which is impossible for $\sigma \in S_n$

Exercise Show similarly

$$\det \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = \det \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ * & & \lambda_n \end{pmatrix} = \prod_{i=1}^{n} \lambda_i$$

Lemma $\det A = \det A^T$.

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proof $\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} a_{\sigma(i)i}$

$j = \sigma(i)$, σ bijection

$$= \sum_{\sigma \in S_n} \underset{\substack{\parallel \\ \varepsilon(\sigma^{-1})}}{\varepsilon(\sigma)} \prod_{i=1}^{n} a_{i\sigma^{-1}(i)}$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma^{-1}) \prod_{i=1}^{n} a_{i\sigma^{-1}(i)} = \overbrace{\sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}}^{\det A^T}$$

$\Rightarrow \det A = \det A^T$ $\square$

Def A *volume form* d on F^n is a function:

$$\underbrace{F^n \times F^n \times \dots \times F^n}_{n} \longrightarrow F \text{ such that:}$$

(i) d multilinear: for any $1 \le i \le n$, $\forall (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n) \in \underbrace{F^n \times \dots \times F^n}_{n-1}$,

$$F^n \longrightarrow F$$
$$v \longmapsto d(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_n)$$

is linear ($\in (F^n)^*$)

(ii) d alternate: if $v_i = v_j$ for some $i \neq j$,

then: $d(v_1, \dots, v_n) = 0$.

I want to show that there is in fact only one such volume form on $F^n \times \dots \times F^n$, and it is given by the determinant.

- $A = (a_{ij}) = \left(A^{(1)} \middle| \dots \middle| A^{(n)} \right)$

column vector 1   column vector n

Lemma $F^n \times \dots \times F^n \longrightarrow F$, $\left(A^{(1)}, \dots, A^{(n)}\right) \longmapsto \det A$ is a volume form

proof (i) Multilinear: Fix $\sigma \in S_n$, then:

$\left(\prod_{i=1}^{n} a_{\sigma(i) i} \right)$ is multilinear: there is only one term from each column appearing in this expression. Then the sum of multilinear maps is multilinear $\rightsquigarrow$ n.

(ii) Alternate $\quad k \neq l, \quad A^{(k)} = A^{(l)}$

Let $\tau \equiv$ permutation which exchanges k and l

$$\begin{pmatrix} 1 & \dots & k & \dots & l & \dots & n \\ \downarrow & & & & & & \downarrow \\ 1 & & l & & k & & n \end{pmatrix} \tau$$

Then : $\boxed{a_{ij} = a_{i\tau(j)}} \quad \forall i, j \in \{1, \dots, n\}$

I can decompose :

$$S_n = \underbrace{A_n}_{\text{even \# transpositions}} \sqcup \tau A_n$$

bijection

disjoint union

$$\det A = \sum_{\sigma \in A_n} \prod_{i=1}^{n} a_{i\sigma(i)} - \sum_{\sigma \in A_n} \prod_{i=1}^{n} \underbrace{a_{i\tau\sigma(i)}}_{= a_{i\sigma(i)}}$$

$$= \sum_{\sigma \in A_n} \prod_{i=1}^{n} a_{i\sigma(i)} - \sum_{\sigma \in A_n} \prod_{i=1}^{n} a_{i\sigma(i)} = 0 .$$

Lemma Let d be a volume form. Then swapping two entries changes the sign:

$$d(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n)$$

$$= - d(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n)$$

proof $d(v_1, \ldots, \boxed{v_i + v_j}, \ldots, \boxed{v_i + v_j}, \ldots, v_n) = 0$

ith position   jth position   alternate

$$= d(v_1, \ldots, v_i, \ldots, v_i, \ldots, v_n) = 0$$

$$+ d(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n)$$

$$+ d(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$$

$$+ d(v_1, \dots, v_j, \dots, v_j, \dots, v_n) = 0$$

$$\Rightarrow \quad d(v_1, \dots, v_i, \dots, v_j, \dots, v_n)$$

$$+ d(v_1, \dots, v_j, \dots, v_i, \dots, v_n) = 0 \qquad \square$$

Cor $\sigma \in S_n$, d volume form:

$$d(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \varepsilon(\sigma)\, d(v_1, \dots, v_n)$$

proof $\sigma = \prod_{i=1}^{n_\sigma} \tau_i$, τ_i = permutation. $\square$

$\boxed{\text{Th}}$ Let d be a volume form on F^n.
Let $A = (A^{(1)} | \dots | A^{(n)})$. Then:

$$d(A^{(1)} | \dots | A^{(n)}) = \det A \, (d(e_1, \dots, e_n))$$

$$e_i = \begin{array}{l} \begin{vmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{vmatrix} \begin{array}{l} \\ \\ \leftarrow i \\ \\ \\ \\ \end{array} \end{array}, \quad 1 \leq i \leq n.$$

$\rightsquigarrow$ Up to a constant, det is the <u>ONLY</u> volume form on F^n.

<u>proof</u> $d(A^{(1)}, \dots, A^{(n)})$

$$= d\left(\sum_{i=1}^{n} a_{i1} e_i, A^{(2)}, \dots, A^{(n)}\right)$$

$$\underset{\text{linearity}}{=} \sum_{i=1}^{n} a_{i1} \, d(e_i, A^{(2)}, \dots, A^{(n)})$$

$$= \sum_{i=1}^{n} a_{i1}\, d\Big(e_i, \sum_{j=1}^{n} a_{j2} e_j, A^{(3)}, \dots, A^{(n)}\Big)$$

$$= \sum_{i,j=1}^{n} a_{i1} a_{j2}\, d\big(e_i, e_j, A^{(3)}, \dots, A^{(n)}\big)$$

linearity

$$= \sum_{\substack{1 \le i_1 \le n \\ 1 \le i_2 \le n \\ \vdots \\ 1 \le i_n \le n}} \prod_{k=1}^{n} a_{i_k k}\, \underbrace{d\big(e_{i_1}, e_{i_2}, \dots, e_{i_n}\big)}$$

For this not to be zero

I need all the i_k's to be distinct:

$$\Longleftrightarrow \exists \sigma \in S_n \ / \ i_k = \sigma(k)$$

$$= \sum_{\sigma \in S_n} \prod_{k=1}^{n} a_{\sigma(k)k}\, d\big(e_{\sigma(1)}, \dots, e_{\sigma(n)}\big)$$

alternate

$$= \sum_{\sigma \in S_n} \prod_{k=1}^{n} a_{\sigma(k)k}\, \varepsilon(\sigma)\, d(e_1, \dots, e_n)$$

$$= d(e_1, \dots, e_n) \left[\sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{k=1}^{n} a_{\sigma(k)k} \right]$$

$$= d(e_1, \dots, e_n) \det A \qquad \square$$

$(*)$

$$\boxed{\textbf{Cor}\quad \det \text{ is the unique volume form such that: } d(e_1, \dots, e_n) = 1.}$$