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# Lecture 11

# DETERMINANTS

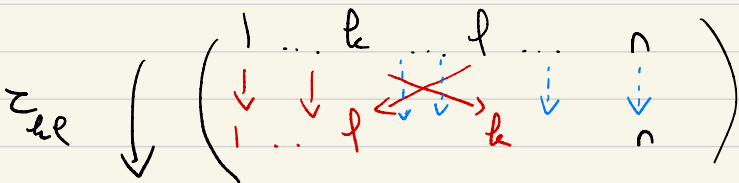
## Permutations and transpositions

• permutation:  $S_n =$  group of permutations of  $\{1, \dots, n\}$

$\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  bijection

$\sigma \equiv$  permutation

• transposition:  $k \neq l, \tau_{kl} \in S_n$



Exchange  $k$   
and  $l$ .

• decomposition any permutation  $\sigma$  can be decomposed  
as a product of transpositions:

$$\sigma = \prod_{i=1}^{n-1} \tau_i, \quad \tau_i \text{ transposition}$$

Signature of a permutation:

$$\epsilon : S_n \rightarrow \{1, -1\}$$

$$\sigma \mapsto \begin{cases} 1 & \text{if } n_\sigma \text{ even} \\ -1 & \text{if } n_\sigma \text{ odd} \end{cases}$$

Then:  $\epsilon$  is a homomorphism:

$$\epsilon(\sigma \circ \tau) = \epsilon(\sigma) \epsilon(\tau)$$

**Def**  $A \in M_n(F)$ ,  $A = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$

We define the determinant of  $A$ :

$$\det A = \sum_{\sigma \in S_n} \epsilon(\sigma) \underbrace{a_{\sigma(1)1} a_{\sigma(2)2} \dots a_{\sigma(n)n}}$$

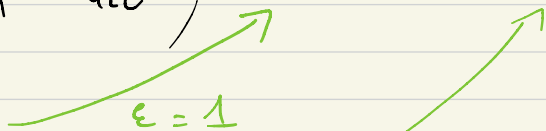
$n!$  summands

one term for each row and column

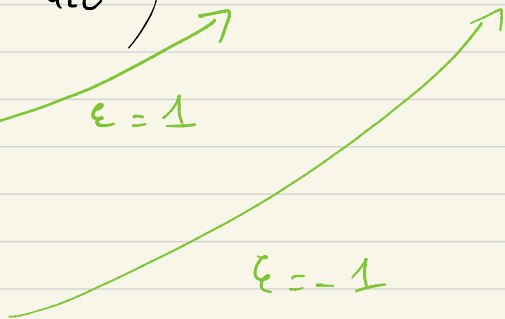
Ex

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

•  $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$



•  $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$



Lemma If  $A = (a_{ij})$  is an upper (lower) triangular matrix:

$$a_{ij} = 0 \text{ for } i \geq j \text{ (resp } i \leq j)$$

Then:  $\det A = 0$ .

$$\det \begin{pmatrix} 0 & & * \\ & \diagdown & \\ 0 & & 0 \end{pmatrix} = = \det \begin{pmatrix} 0 & & 0 \\ & \diagdown & \\ * & & 0 \end{pmatrix} = 0$$

proof  $\det A = \sum_{\sigma \in S_n} \epsilon(\sigma) a_{\sigma(1)1} \dots a_{\sigma(n)n}$

For the summand not to be zero, we need:

$$\sigma(j) < j \quad \forall j \in \{1, \dots, n\}$$

which is impossible for  $\sigma \in S_n$  □

Example Show similarly

$$\det \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = \det \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ * & & \lambda_n \end{pmatrix} = \prod_{i=1}^n \lambda_i$$

Lemma  $\det A = \det A^T$

proof  $\det A = \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^n a_{\sigma(i)i}$

$$= \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^n a_{i\sigma^{-1}(i)}$$

$j = \sigma(i)$ ,  $\sigma$  bijection

$\det A^T$

$$= \sum_{\sigma \in S_n} \epsilon(\sigma^{-1}) \prod_{i=1}^n a_{i\sigma^{-1}(i)} = \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^n a_{i\sigma(i)}$$

$$\Rightarrow \det A = \det A^T$$

□

Def A volume form  $d$  on  $F^n$  is a function:

$$\underbrace{F^n \times F^n \times \dots \times F^n}_n \rightarrow F \text{ such that:}$$

(i)  $d$  multilinear: for any  $1 \leq i \leq n$ ,  $\forall$

$$(\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n) \in \underbrace{F^n \times \dots \times F^n}_{n-1},$$

$$F^n \rightarrow F$$

$$\sigma \mapsto d(\sigma_1, \dots, \sigma_{i-1}, \sigma, \sigma_{i+1}, \dots, \sigma_n)$$

is linear ( $\in (F^n)^*$ )

(ii)  $d$  alternate: if  $\sigma_i = \sigma_j$  for some  $i \neq j$ ,

$$\text{Then: } d(\sigma_1, \dots, \sigma_n) = 0.$$

I want to show that there is in fact only one such volume form on  $F^{\wedge} \times \dots \times F^{\wedge}$ , and it is given by the determinant.

$$A = (a_{ij}) = \left( \begin{array}{c|c|c} A^{(1)} & \dots & A^{(n)} \end{array} \right)$$

$\uparrow$   
column vector  
 $\downarrow$   
1
 $\uparrow$   
column vector  
 $\downarrow$   
n

lemma  $F^{\wedge} \times \dots \times F^{\wedge} \rightarrow F$  is a  
 $(A^{(1)}, \dots, A^{(n)}) \mapsto \det A$  volume form

proof (i) Multi-linear Fix  $\sigma \in S_n$ , then:

$\left( \prod_{i=1}^n a_{\sigma(i)i} \right)$  is multi-linear: there is only one term from each column appearing in this

expression. Then the num of multilinear maps is  
 multilinear  $\rightarrow n$ .

(ii) Alternate  $k \neq l, A^{(k)} = A^{(l)}$

let  $\tau \equiv$  permutation which exchanges  
 $k$  and  $l$

$$\left( \begin{array}{cccccc} 1 & \dots & k & \dots & l & \dots & n \\ \downarrow & & \swarrow & \searrow & \downarrow & & \\ \color{red}{1} & & \color{red}{l} & \color{red}{\rightarrow} & \color{red}{k} & & \color{red}{n} \end{array} \right) \tau$$

Then:  $a_{ij} = a_{i\tau(j)} \quad \forall i, j \in \{1, \dots, n\}$

I can decompose:

$$S_n = A_n \sqcup \tau A_n$$

even #  
 transpositions

bijection

disjoint union  $n$

$$\det A = \sum_{\sigma \in A_n} \prod_{i=1}^n a_{i\sigma(i)} = \sum_{\sigma \in A_n} \prod_{i=1}^n \underbrace{a_{i\tau\sigma(i)}}_{= a_{i\sigma(i)}}$$



$$= \sum_{\sigma \in A_n} \prod_{i=1}^n a_{i\sigma(i)} - \sum_{\sigma \in A_n} \prod_{i=1}^n a_{i\sigma(i)} = 0.$$

lemma let  $d$  be a volume form. Then swapping two entries changes the sign:

$$d(v_1, \dots, v_i, \dots, v_j, \dots, v_n) \\ = -d(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$$

proof

$$d(v_1, \dots, \boxed{v_i + v_j}, \dots, \boxed{v_i + v_j}, \dots, v_n) = 0$$

↑  
i<sup>th</sup> position
↑  
j<sup>th</sup> position
↑  
alternate

$$= d(v_1, \dots, v_i, \dots, v_i, \dots, v_n) + d(v_1, \dots, v_i, \dots, v_j, \dots, v_n)$$

$$+ d(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$$

$$+ d(v_1, \dots, v_j, \dots, v_j, \dots, v_n) = 0$$

$$\Rightarrow d(v_1, \dots, v_i, \dots, v_j, \dots, v_n)$$

$$+ d(v_1, \dots, v_j, \dots, v_i, \dots, v_n) = 0$$

□

Cor  $\sigma \in S_n$ ,  $d$  volume form:

$$d(g_{\sigma(1)}, \dots, g_{\sigma(n)}) = \varepsilon(\sigma) d(g_1, \dots, g_n)$$

proof  $\sigma = \prod_{i=1}^{n_\sigma} \tau_i$ ,  $\tau_i = \text{permutation}$ .

□

**Th** Let  $d$  be a volume form on  $F^n$ .  
 Let  $A = (A^{(1)} | \dots | A^{(n)})$ . Then:

$$d(A^{(1)} | \dots | A^{(n)}) = \det A (d(e_1, \dots, e_n))$$

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \leftarrow i \\ \vdots \\ 0 \end{pmatrix}, \quad 1 \leq i \leq n.$$

$\leadsto$   $U_p$  is a constant,  $\det$  is the ONLY volume form on  $F^n$ .

proof  $d(A^{(1)}, \dots, A^{(n)})$

$$= d\left(\sum_{i=1}^n a_{i1} e_i, A^{(2)}, \dots, A^{(n)}\right)$$

$$= \sum_{i=1}^n a_{i1} d(e_i, A^{(2)}, \dots, A^{(n)})$$

$\uparrow$   
 linearity

$$= \sum_{i=1}^n a_{i1} d(e_i, \sum_{j=1}^n a_{j2} e_j, A^{(3)}, \dots, A^{(n)})$$

$$= \sum_{i,j=1}^n a_{i1} a_{j2} d(e_i, e_j, A^{(3)}, \dots, A^{(n)})$$

linearity

$$= \sum_{\substack{1 \leq i_1 \leq n \\ 1 \leq i_2 \leq n \\ \vdots \\ 1 \leq i_n \leq n}} \prod_{k=1}^n a_{i_k k} d(e_{i_1}, e_{i_2}, \dots, e_{i_n})$$

for this not to be zero,

I need all the  $i_k$ 's to be distinct:

$$\Leftrightarrow \exists \sigma \in S_n / i_k = \sigma(k)$$

$$= \sum_{\sigma \in S_n} \prod_{k=1}^n a_{\sigma(k)k} d(e_{\sigma(1)}, \dots, e_{\sigma(n)})$$

alternate

$$= \sum_{\sigma \in S_n} \prod_{k=1}^n a_{\sigma(k)k} e(\sigma) d(e_1, \dots, e_n)$$

$$= d(e_1, \dots, e_n) \left[ \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{k=1}^n a_{\sigma(k)k} \right]$$

$$= d(e_1, \dots, e_n) \det A$$



Cor  $\det$  is the unique volume form such that:  
 $d(e_1, \dots, e_n) = 1$ .