

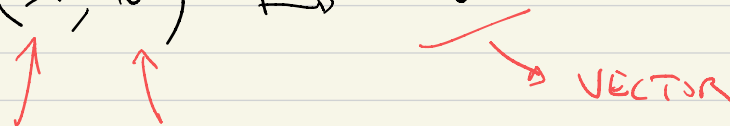

Lecture 1

Vector spaces, subspaces

Let F be an arbitrary field ($F = \mathbb{R}$ or $\mathbb{C}$)

Def (F vector space) An F -vector space (a vector space over F) is an abelian group $(V, +)$ equipped with a function

$$F \times V \rightarrow V$$
$$(\lambda, v) \mapsto \lambda v$$



SCALAR VECTORS

such that :

- $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$
- $(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$
- $\lambda(\mu v) = (\lambda\mu)v$
- $1 \cdot v = v$

→ We know how to:

- sum two vectors
- multiply a vector $v \in V$ by a scalar $\lambda \in F$.

Examples

(i) $n \in \mathbb{N}$, F^n : column vectors of length n with entries in F

$$v \in F^n, \quad v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i \in F, 1 \leq i \leq n.$$

$$\cdot v + w = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}$$

$$\cdot \lambda v = \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix}, \quad \lambda \in F.$$

$\overline{F}^n$ is an F -vector space.

(ii). Any set X ,

$$\mathbb{R}^X = \{ f : X \rightarrow \mathbb{R} \}$$

(set of real valued functions on X)

$\mathbb{R}^X$ $\mathbb{R}$ vector space :

$$\cdot (f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$\cdot (\lambda f)(x) = \lambda f(x), \quad \lambda \in \mathbb{R}$$

(iii) $M_{n,m}(F)$ $\equiv$ $n \times m$ F -valued
matrices

$$\begin{matrix} & & m \\ & \langle & \\ n & \left(\begin{array}{c} a_{ij} \end{array} \right) & \\ & \rangle & \end{matrix} \quad \begin{matrix} a_{ij} \in F \\ | \quad 1 \leq i \leq n \\ | \quad 1 \leq j \leq m \end{matrix}$$

Check as an exercise that the above examples are vector spaces.

Pr The axiom of scalar multiplication imply that: $\forall v \in V, 0 \cdot v = 0$

Def (Subspace) Let V be a vector space over F . The subset U of V is a vector subspace of V (noted $U \leq V$)

if: $0 \in U$

$\left[\begin{array}{l} \cdot (u_1, u_2) \in U \times U \Rightarrow u_1 + u_2 \in U \\ \cdot (\lambda, u) \in F \times U \Rightarrow \lambda u \in U \end{array} \right]$

$\Leftrightarrow \forall (\lambda_1, \lambda_2) \in F \times F, \forall (v_1, v_2) \in U \times U, \lambda_1 v_1 + \lambda_2 v_2 \in U.$

⇒ This property means that U is stable by :

- scalar multiplication
- vector addition

Exercise V F vector subspace
| $U \leq V$

⇒ U F vector subspace.

Examples

① $V = \mathbb{R}^{\mathbb{R}}$ space of functions $\mathbb{R} \rightarrow \mathbb{R}$
 $\mathcal{C}(\mathbb{R}) \leq V$ — $\mathcal{C}^\infty$ functions $\mathbb{R} \rightarrow \mathbb{R}$
 $\mathbb{P}(\mathbb{R}) \leq \mathcal{C}(\mathbb{R})$ space of polynomials

② $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3, x_1 + x_2 + x_3 = 1 \right\}$

Check that this is a subspace for $t=0$ only.

Prop

let V be an F vector space.

let $U, W \leq V$. Then:

$$U \cap W \leq V.$$

proof. $\begin{matrix} 0 \in U \\ 0 \in W \end{matrix} \mid \Rightarrow 0 \in U \cap W$

Stability $(\lambda_1, \lambda_2) \in F^2$
 $(v_1, v_2) \in (U \cap W)^2$

$$\left| \begin{array}{l} \lambda_1 \underbrace{v_1}_{\in U} + \lambda_2 \underbrace{v_2}_{\in U} \in U \\ \lambda_1 \underbrace{v_1}_{\in U} + \lambda_2 \underbrace{v_2}_{\in U} \in W \end{array} \right| \Rightarrow \lambda_1 v_1 + \lambda_2 v_2 \in U \cap W$$



The union of two subspaces is
generally NOT a subspace

→ it is typically not stable by addition

$$\underline{E_x} \quad V = \mathbb{R}^2$$



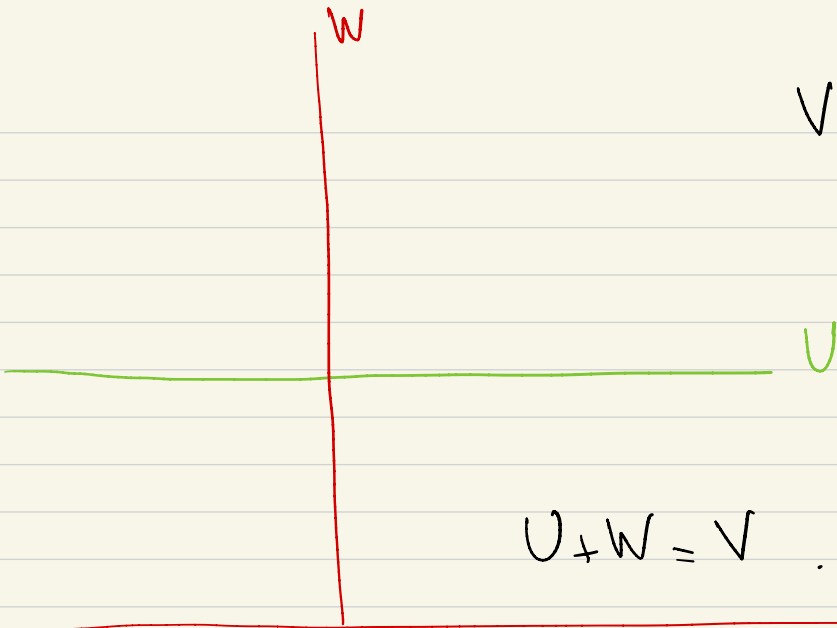
Def (Sum of subspaces) let V be an
 F vector space let $U \leq V$ $W \leq V$

The sum of U and W is the set:

$$U + W = \{u + w, (u, w) \in U \times W\}$$

Ex.

$$V = \mathbb{R}^2$$



$$U + W = V$$

Prop

$$V \text{ } F \text{ vector space} \\ (U \leq V, W \leq V) \Rightarrow U + W \leq V.$$

proof . $0 = \underbrace{0}_{\in U} + \underbrace{0}_{\in W} \in U + W$

$$\lambda_1 f + \lambda_2 g, \quad \begin{array}{l} f \in U + W, g \in U + W \\ f = \underbrace{f_1}_{\in U} + \underbrace{f_2}_{\in W}, \quad g = \underbrace{g_1}_{\in U} + \underbrace{g_2}_{\in W} \end{array}$$

$$\Rightarrow \lambda_1 f + \lambda_2 g$$

$$= \lambda_1 (f_1 + f_2) + \lambda_2 (g_1 + g_2)$$

$$= \underbrace{\left(\lambda_1 \underbrace{f_1}_{\in U} + \lambda_2 \underbrace{g_1}_{\in U} \right)}_{\substack{\Downarrow U \subseteq V \\ \in U}} + \underbrace{\left(\lambda_1 \underbrace{f_2}_{\in W} + \lambda_2 \underbrace{g_2}_{\in W} \right)}_{\substack{\Downarrow W \subseteq V \\ \in V}}$$

$$\Downarrow U \subseteq V$$

+

$$\Downarrow W \subseteq V$$

$$\in U + W$$

□

Exercise Show that $U + W$ is the smallest subspace of V which contains U and W .

Subspaces and quotient

Def (Quotient) Let V be an F vector space, let $U \leq V$. The quotient space V/U is the abelian group V/U equipped with the scalar multiplication:

$$\begin{aligned} F \times V/U &\longrightarrow V/U \\ (\lambda, v + U) &\longmapsto \lambda v + U \end{aligned} \quad (*)$$

Prop V/U is an F vector space.

Check that the multiplication operation $(*)$ is well defined. Indeed,

$$v_1 + U = v_2 + U \quad \text{then:}$$

$$\Rightarrow v_1 - v_2 \in U$$

$$\Rightarrow \lambda(v_1 - v_2) \in U$$

$$\begin{array}{c} \uparrow \\ U \subseteq V \end{array} \Rightarrow \lambda v_1 + U = \lambda v_2 + U \in V/U$$

$\Rightarrow (*)$ is well defined.

Exercise Check that V/U is an F vector space.

Next lecture $\sim$ independence of vectors.