Lecture 7 Elementary operations and elementary matrices

Special case of the change of basis formula

- $\alpha : V \longrightarrow W$ linear

  $(\underline{B}, \underline{B}')$ basis of V

  $(\underline{C}, \underline{C}')$ basis of W

  $$[\alpha]_{\underline{B}\,\underline{C}} \rightsquigarrow [\alpha]_{\underline{B}'\,\underline{C}'}$$

- $\underline{V = W}$ $\quad \alpha : V \longrightarrow V$ linear

  ($\equiv$ endomorphism)

  - $\underline{B} = \underline{C}$ $\quad \underline{B}' = \underline{C}'$
  - $P =$ change of matrix from $\underline{B}'$ to $\underline{B}$

<u>then</u> $[\alpha]_{B', B'} = P^{-1} [\alpha]_{B, B} P$

Def A, A' $n \times n$ (square) matrices.
We say that A and A' are <u>similar</u> ($\equiv$ conjugate) iff:

$$\begin{cases} A' = P^{-1} A P \\ P \equiv n \times n \text{ square invertible} \end{cases}$$

$\longrightarrow$ Central concept when dealing with <u>diagonalization</u> of endomorphisms

$\rightsquigarrow$ SPECTRAL THEORY.

Elementary operations and elementary matrices

Def Elementary column operation on an $m \times n$ matrix A :

(i) swap columns i and j $(i \neq j)$

(ii) replace column i by $\lambda \times$ column i
$(\lambda \neq 0, \lambda \in F)$

(iii) add $\lambda \times$ column i to column j $(i \neq j)$

- Elementary row operations : analogous way
- These elementary operations are invertible.
- These operations can be realized through

The action of elementary matrices :

(i) i, j , $i \neq j$

$$T_{ij} = \left(\begin{array}{ccccccccccc}
1 &&&&&&&&&& \\
& \ddots &&&&&&&&& \\
&& 1 &&&&&&&& \\
&&& 0 & \cdots & \cdots & \cdots & 1 &&& \\
&&& \vdots & 1 &&& \vdots &&& \\
&&& \vdots && \ddots && \vdots &&& \\
&&& \vdots &&& 1 & \vdots &&& \\
&&& 1 & \cdots & \cdots & \cdots & 0 &&& \\
&&&&&&&& 1 && \\
&&&&&&&&& \ddots & \\
&&&&&&&&&& 1
\end{array}\right)
\begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{array}$$

(ii)

$$M_{i,\lambda} = \left(\begin{array}{ccccccc}
1 &&&&&& \\
& \ddots &&& 0 && \\
&& 1 &&&& \\
&&& \lambda &&& \\
&&&& 1 && \\
& 0 &&&& \ddots & \\
&&&&&& 1
\end{array}\right)
\begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{array}$$

(iii) $C_{i,j,\lambda} = Id + \lambda E_{ij}$

$$E_{ij} = \begin{array}{cc} \begin{array}{ccc} & j & \end{array} & \\ \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} & \begin{array}{c} \\ i \\ \\ \end{array} \end{array}$$

Links between elementary operations and matrices:

An elementary column (row) operation can be performed by multiplying A by the corresponding elementary matrix from the right (left).

Exercise

Ex: $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$$

Constructive proof that any $m \times n$ matrix is equivalent to:

$$\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) \quad \text{for some } r.$$

- Start with A. If all entries are zero, done.
- Pick $a_{ij} = \lambda \neq 0$:
  - swap rows i and 1
  - swap columns j and 1

$\Rightarrow$ get λ in position $(1,1)$

- Multiply column 1 by $1/\lambda$ $(\lambda \neq 0)$

$\Rightarrow$ get 1 in position $(1,1)$

- Now clear out row 1 and column 1 using elementary operations of type (iii) :

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \tilde{A} & \\ 0 & & & \end{pmatrix}$$

- Continue with $\tilde{A}$ $(m-1)\times(n-1)$.
- End of the process :

$$\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) = Q^{-1} A P.$$

$$= \underbrace{\overbrace{E_\ell^{-1} \cdots E_1^{-1}}^{Q^{-1}}}_{\text{row operations}} A \underbrace{\overbrace{E_1 \cdots E_c}^{P}}_{\text{Column operations}}$$

Variations

① Gauss' pivot algorithm

If you use only row operations, you can reach the so called "row echelon form" of the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & b \\ & & 1 & 0 & c \\ & 0 & & 1 & d \end{pmatrix}$$

"pivot"

- Assume that $a_{i1} \neq 0$ for some i
- Swap rows i and 1
- Divide first row by $\lambda = a_{i1}$ to get 1 in $(1,1)$
- Use 1 to clean the rest of the 1st column

- move to 2d column
- iterate

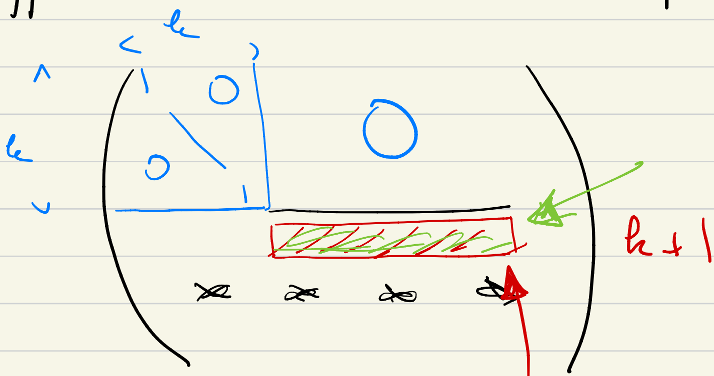
$\Rightarrow$ This procedure is exactly what you do when solving a linear system of equations (Gauss' pivot algorithm)

② Representation of square invertible matrices.

Lemma If A is a $n \times n$ (square) matrix, then we can obtain I_n using row elementary operations only (resp. column operations only)

proof We do the proof for column operations.
We argue by induction on the number of rows.

. Suppose that we could reach the form:



. I want to obtain the same structure with $k+1$ rows.

. **Claim** $\exists\, j > k \ / \ a_{k+1,j} = \lambda > 0$.

Indeed, otherwise we can show that:

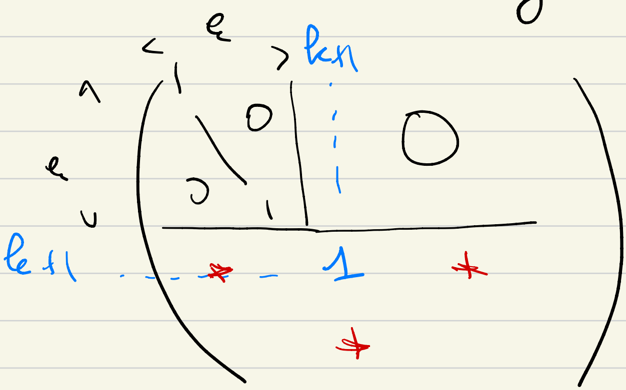
$$\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}\begin{matrix}\\ \\ \\ \leftarrow k+1\\ \\ \\ \\ \end{matrix}$$

NOT in the span of the column vectors of A.

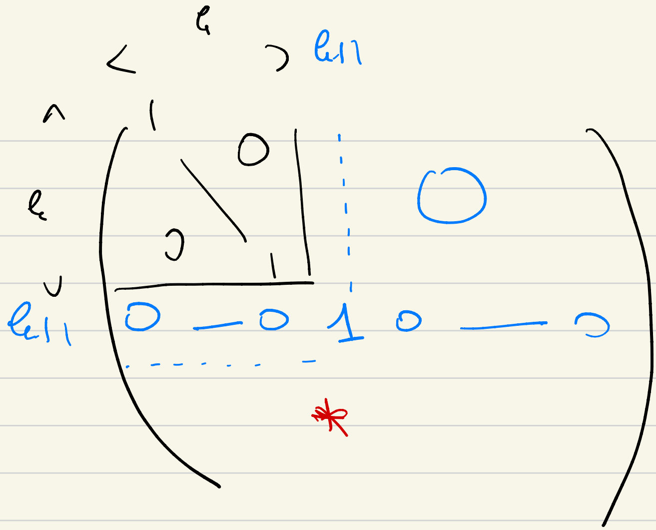
(this is an exercise)

This contradicts our assumption that A is invertible.

- swap column $k+1$ and j
- divide column $k+1$ by $\lambda = a_{k+1,j} \neq 0$



- use 1 to clear the rest of the $(k+1)$ row using type (iii) elementary operations.



$\rightsquigarrow$ structure for the induction hypothesis at the order $k+1$.

Outcome $AE_1 \dots E_c = I_n$

$\Rightarrow \quad A^{-1} = E_1 \dots E_c$

$\Rightarrow$ algorithm for computing A^{-1}

($\Leftrightarrow$ algorithm for solving the linear system of n unknowns with n equations :

$$AX = F \iff X = A^{-1}F$$)

$\rightsquigarrow$ Constructive way to compute the inverse of A.

Prop Any invertible square matrix is a product of elementary matrices.