


lecture 4 linear maps, isomorphisms and the rank-nullity Theorem

Def (linear map) V, W are $\mathbb{F}$ -vector spaces.
A map $\alpha: V \rightarrow W$ is linear iff:

$$\forall (\lambda_1, \lambda_2) \in \mathbb{F}^2, \forall (v_1, v_2) \in V \times V,$$

$$\alpha(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2)$$

Example

(i)

$$n = \bigwedge_{\substack{1 \\ \vdots \\ n}}^m \left(\begin{matrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{matrix} \right) \begin{matrix} \wedge \\ \vdots \\ \wedge \end{matrix} \begin{matrix} m \\ \\ \\ \\ \end{matrix}$$

$$\alpha: \mathbb{R}^m \rightarrow \mathbb{R}^n \quad \text{linear.}$$

$$x \mapsto nx$$

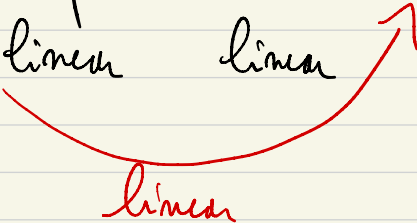
(ii) $\alpha : \mathcal{C}([0;1]) \longrightarrow \mathcal{C}([0;1])_n$
 $f \longmapsto \alpha(f)(n) = \int_0^1 f(t) dt$

linear map.

(iii) Fix $x \in [a;b]$, $\mathcal{C}([a;b]) \longrightarrow \mathbb{R}$
 $f \longmapsto f(x)$
 linear map.

Remark U, V, W $\mathbb{F}$ vector spaces

(i) $\text{Id}_V : V \longrightarrow V$ linear map
 $x \longmapsto x$

(ii) $U \xrightarrow{\beta} V \xrightarrow{\alpha} W$
 β linear α linear


linearity is
 stable by
 composition

Lemma V, W F vector spaces.

B basis for V .

If $\alpha_0: B \rightarrow W$ is ANY map, then there is a unique linear map $\alpha: V \rightarrow W$ extending α_0 (ie $\forall v \in B, \alpha_0(v) = \alpha(v)$)

proof $v \in V, \quad v = \sum_{i=1}^n \lambda_i v_i$
 $B = (v_1, \dots, v_n)$

Necessarily by linearity:

$$\alpha(v) = \alpha\left(\sum_{i=1}^n \lambda_i v_i\right) = \sum_{i=1}^n \lambda_i \alpha(v_i)$$

Remark (i) True for ∞ dimensional spaces as well.
Often to define a linear map, we define its value on a basis and "extend by linearity".

(ii) $\alpha_1, \alpha_2: V \rightarrow W$ linear. If they agree on a basis $\mathcal{B}$ of V , they are equal.

Def (Isomorphism) V, W vector spaces over F .

A map $\alpha: V \rightarrow W$ is called an isomorphism iff:

(i) α linear

(ii) α bijection

If such an α exists, we note: $V \cong W$
(V is isomorphic to W)

Pr

$\alpha: V \rightarrow W$ isomorphism

$\Rightarrow \alpha^{-1}: W \rightarrow V$ is linear.

Indeed, $\alpha: V \rightarrow W$

$\alpha^{-1}: W \rightarrow V$

$(w_1, w_2) \in W \times W, \quad w_1 = \alpha(v_1), \quad w_2 = \alpha(v_2)$

$$\begin{aligned} & \alpha^{-1}(w_1 + w_2) \\ &= \alpha^{-1}(\alpha(v_1) + \alpha(v_2)) \\ &= \alpha^{-1}(\alpha(v_1 + v_2)) = v_1 + v_2 \\ &= \alpha^{-1}(w_1) + \alpha^{-1}(w_2). \end{aligned}$$

Similarly, $\forall (\lambda, w) \in F \times W$

$$\alpha^{-1}(\lambda w) = \lambda \alpha^{-1}(w).$$

Lemma $\simeq$ is an equivalence relation on the class of all vector spaces over F .

(i) $\text{id}_V: V \rightarrow V$ isomorphism

(ii) $\alpha: V \rightarrow W$ isomorphism $\Rightarrow \alpha^{-1}: W \rightarrow V$ isomorphism

(iii) If $U \xrightarrow{\beta} V \xrightarrow{\alpha} W$

$\underbrace{\beta}_{\text{isomorphic}}$

$\Rightarrow U \xrightarrow{\alpha \circ \beta} W$

$\underbrace{\alpha \circ \beta}_{\text{isomorphic}}$

proof Exercise.

Thm If V is a vector space over F of dimension n then:

$$V \cong F^n$$

$$\left(F^n = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i \in F, 1 \leq i \leq n \right\} \right)$$

proof let $B = (v_1, \dots, v_n)$ be a basis of V .

Then:

$$\alpha: V \longrightarrow F^n$$
$$v = \sum_{i=1}^n \lambda_i v_i \mapsto \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

is an isomorphism,

$\rightarrow$ exercise.

Def Choosing a basis of V is like choosing an isomorphism from V to F^n .

Thm let V, W F vector spaces with finite dimension, Then $V \cong W$ iff they have the same dimension.

proof $\Leftarrow$ $\dim V = \dim W = n$

$$\Rightarrow \begin{cases} V \simeq F^n \\ W \simeq F^n \end{cases} \Rightarrow V \simeq W.$$

$\Rightarrow$ $\alpha: V \rightarrow W$ isomorphism, B is a basis for V , then:

claim $\alpha(B)$ basis for W .

- $\alpha(B)$ spans W follows from surjectivity of α
- $\alpha(B)$ free family ——— injectivity of α

$\rightarrow$ exercise.

Def (Kernel and image of a linear map)

Let V, W vector spaces over F .

Let $\alpha: V \rightarrow W$ linear map.

We define :

$$\bullet N(\alpha) = \text{Ker } \alpha = \{v \in V \mid \alpha(v) = 0\}$$

($\equiv$ Kernel of α)

$$\bullet \text{Im}(\alpha) = \{w \in W \mid \exists v \in V, w = \alpha(v)\}$$

($\equiv$ Image of α)

Lemma $\text{Ker } \alpha$ and $\text{Im } \alpha$ are subspaces of
respectively V and W .

proof $(\lambda_1, \lambda_2) \in F^2, (v_1, v_2) \in \text{Ker } \alpha \times \text{Ker } \alpha,$

$$\begin{aligned} \alpha(\lambda_1 v_1 + \lambda_2 v_2) &= \lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2) \\ &\quad \quad \quad \downarrow \quad \quad \quad \downarrow \\ &\quad \quad \quad 0 \quad \quad \quad 0 \\ &= 0 \end{aligned}$$

$$\Rightarrow \lambda_1 v_1 + \lambda_2 v_2 \in \text{Ker } \alpha.$$

$$\cdot (\lambda_1, \lambda_2) \in \mathbb{F}^2, \quad (w_1, w_2) \in (\text{Im } \alpha)^2, \\ w_1 = \alpha(v_1), \quad w_2 = \alpha(v_2)$$

$$\Rightarrow \lambda_1 w_1 + \lambda_2 w_2 = \lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2) \\ = \alpha(\lambda_1 v_1 + \lambda_2 v_2) \\ \in \text{Im } \alpha \quad \text{D.}$$

Ex $\alpha: \mathcal{C}^\infty(\mathbb{R}) \rightarrow \mathcal{C}^\infty(\mathbb{R})$

$$f(t) \mapsto \alpha(f)(t) = f''(t) + f(t)$$

$$\cdot \text{Ker } \alpha = \{f \in \mathcal{C}^\infty(\mathbb{R}), f'' + f = 0\} \\ = \text{Span}_{\mathbb{R}} \langle e^t, e^{-t} \rangle$$

$$\cdot \text{Im } \alpha = ?$$

Th

let V, W F vector spaces.

let $\alpha: V \rightarrow W$ linear map. Then:

$$V / \text{Ker } \alpha \xrightarrow{\bar{\alpha}} \text{Im } \alpha$$

$$\bar{\alpha}(v + \text{Ker } \alpha) \mapsto \alpha(v)$$

is an isomorphism.

proof . $\bar{\alpha}$ well defined: $v + \text{Ker } \alpha = v' + \text{Ker } \alpha$

$$\Rightarrow v - v' \in \text{Ker } \alpha \Rightarrow$$

$$\alpha(v - v') = 0 \Rightarrow \alpha(v) = \alpha(v').$$

$$\Rightarrow \bar{\alpha}(v) = \bar{\alpha}(v')$$

• $\bar{\alpha}$ linear follows immediately from α linear.

• $\bar{\alpha}$ bijection:

- injectivity: $\bar{\alpha}(v + \text{Ker } \alpha) = 0$

$$\Rightarrow \alpha(v) = 0 \Rightarrow v \in \text{Ker } \alpha \Rightarrow$$

$$v + \text{Ker } \alpha = 0 + \text{Ker } \alpha.$$

- surjectivity follows from the definition of $\text{Im } \alpha$:

$$w \in \text{Im } \alpha, \exists v \in V / w = \alpha(v) = \bar{\alpha}(v).$$

Def (Rank and nullity)

$$\bullet r(\alpha) = \text{rk}(\alpha) = \dim \text{Im}(\alpha) \quad (\text{rank})$$

$$\bullet n(\alpha) = \dim \text{Ker}(\alpha) \quad (\text{nullity})$$

Th (Rank-nullity Theorem)

• let U, V be vector spaces over F ,
 $\dim_F U < +\infty$.

Let $\alpha: V \rightarrow V$ be a linear map, then:

$$\dim U = r(\alpha) + n(\alpha)$$

proof We have proved that:

$$U / \text{Ker } \alpha \cong \text{Im } \alpha$$

$$\Rightarrow \dim(U / \text{Ker } \alpha) = \dim \text{Im } \alpha$$

$$\parallel$$
$$\dim U - \dim \text{Ker } \alpha$$

$$\Rightarrow \dim U = r(\alpha) + n(\alpha)$$

Q.E.D.

Lemma (Characterization of isomorphism)

V, W vector spaces over F of **EQUAL FINITE DIMENSION**. Let $\alpha: V \rightarrow W$

linear map, then TFAE:

(i) α injective

(ii) α surjective

(iii) α isomorphism

proof $\rightarrow$ exercise. Follows directly from the rank-nullity theorem.

$$\underline{\underline{Ex}} \quad \left| \begin{array}{l} V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x+y+z=0 \right\} \\ \dim V = ? \end{array} \right.$$

$$\alpha: \mathbb{R}^3 \rightarrow \mathbb{R}$$

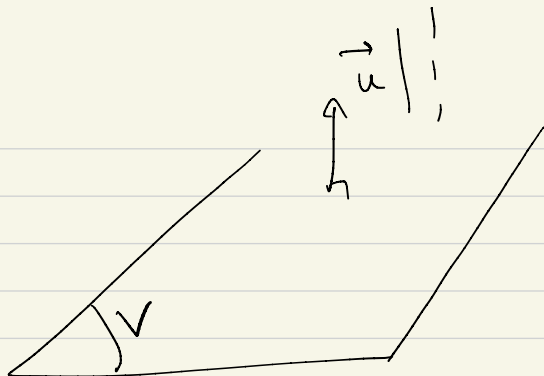
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto x+y+z$$

$$\ker \alpha = V$$

$$\operatorname{Im} \alpha = \mathbb{R}$$

$$\Rightarrow 3 = r(\alpha) + 1, \quad r(\alpha) = 2.$$

(V is a plane)



0.