


Lecture 20

Hermitian form / $\mathbb{C}$

Skew symmetric form / $\mathbb{R}$

φ sesquilinear $V \times V \rightarrow \mathbb{C}$
| $\dim_{\mathbb{C}} V$

• linear first variable $\varphi(\lambda u, v) = \lambda \varphi(u, v)$

• antilinear second variable $\varphi(u, \lambda v) = \overline{\lambda} \varphi(u, v)$

Def (Hermitian form) A sesquilinear form $\varphi: V \times V \rightarrow \mathbb{C}$ is called Hermitian if:

$$\forall (u, v) \in V \times V, \quad \varphi(u, v) = \overline{\varphi(v, u)}$$

Remark φ Hermitian, $\varphi(u, u) = \overline{\varphi(u, u)}$

$$\Rightarrow \varphi(u, u) \in \mathbb{R}$$

$$\text{Lemma: } \varphi(\lambda u, \lambda u) = |\lambda|^2 \varphi(u, u)$$

$\leadsto$ We may talk about positive / negative (semi) definite Hermitian form.

Lemma

A sesquilinear form $\varphi: V \times V \rightarrow \mathbb{C}$ is Hermitian iff: for **any** basis $\mathcal{B}$ of V ,

$$[\varphi]_{\mathcal{B}} = \overline{[\varphi]_{\mathcal{B}}^T}$$

proof

$$A = [\varphi]_{\mathcal{B}} = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$$

$$a_{ij} = \varphi(e_i, e_j)$$

$$\hookrightarrow \begin{cases} a_{ij} = \varphi(e_i, e_j) \\ a_{ji} = \varphi(e_j, e_i) = \overline{\varphi(e_i, e_j)} = \overline{a_{ij}} \end{cases}$$

↑
Hermitian

$$\Rightarrow [\varphi]_{\mathcal{B}}^T = \overline{[\varphi]_{\mathcal{B}}}$$

Conversely $[\varphi]_{\mathcal{B}} = A, \quad A = \overline{A}^T$

$$\begin{cases} u = \sum_{i=1}^n \lambda_i e_i \\ v = \sum_{i=1}^n \mu_i e_i \end{cases} \quad \mathcal{B} = (e_1, \dots, e_n)$$

$$\begin{aligned} \varphi(u, v) &= \varphi\left(\sum_{i=1}^n \lambda_i e_i, \sum_{j=1}^n \mu_j e_j\right) \\ &= \sum_{i,j=1}^n \lambda_i \overline{\mu_j} \varphi(e_i, e_j) \\ &= \sum_{i,j=1}^n \lambda_i \overline{\mu_j} a_{ij} \end{aligned}$$

$$\begin{aligned}
 \overline{\varphi(v, u)} &= \overline{\varphi\left(\sum_{i=1}^n \mu_i e_i, \sum_{j=1}^n \lambda_j e_j\right)} \\
 &= \sum_{i,j=1}^n \overline{\mu_i \lambda_j} \overline{\varphi(e_i, e_j)} \\
 &= \sum_{i,j=1}^n \overline{\mu_i} \overline{\lambda_j} \overline{a_{ij}} = \sum_{i,j=1}^n \overline{\mu_j} \overline{\lambda_i} \underbrace{\overline{a_{ji}}}_{\substack{= \\ a_{ij}}} \\
 &= \sum_{i,j=1}^n \lambda_i \overline{\mu_j} a_{ij} = \varphi(u, v)
 \end{aligned}$$

Polarisation identity A Hermitian form φ on a complex vector space V is entirely determined by :

$$\begin{aligned}
 Q: V &\rightarrow \mathbb{R} && \text{via the formula:} \\
 v &\mapsto \varphi(v, v)
 \end{aligned}$$

$$\varphi(u, v) = \frac{1}{4} \begin{bmatrix} Q(u+v) - Q(u-v) + iQ(u+iv) \\ -iQ(u-iv) \end{bmatrix} \quad \leftarrow$$

→ Exercise Check the polarisation identity.

Thm (Hermitian formulation of Sylvester's Law)

V be a finite dim. vector space over $\mathbb{C}$.
 $\dim_{\mathbb{C}} V = n$.

let $\varphi: V \times V \rightarrow \mathbb{C}$ be a Hermitian form on V .
 Then $\exists B = (v_1, \dots, v_n)$ of V s.t.:

$$[\varphi]_B = \begin{pmatrix} I_p & & 0 \\ & -I_q & \\ 0 & & 0 \end{pmatrix} \quad \leftarrow$$

where p, q depend only on φ .

proof (sketch: nearly identical to the real case)

. Existence $\varphi \neq 0$, done. Assume not, then by the polarization identity, there exists $e_1 \neq 0$ such that $\varphi(e_1, e_1) \neq 0$.

Rescale to:
$$v_1 = \frac{e_1}{\sqrt{|\varphi(e_1, e_1)|}}$$

so:
$$\underbrace{\varphi(e_1, e_1)}_{\in \mathbb{R}} = \pm 1.$$

Then we consider the orthogonal:

$$W = \{w \in V \mid \varphi(v_1, w) = 0\}$$

and we check (verbatim like in the real case) :

$$V = \langle v_1 \rangle \oplus W \quad (\dim W = n - 1)$$

$\Rightarrow$ Now argue by induction on the dimension to diagonalize $\varphi|_W$

- Uniqueness of p : p is the maximal dimension of a subspace on which φ is definite positive ($\varphi(u, u) \in \mathbb{R}$)
(verbatim like in the real case)
- Similarly for q .

n.

$$F = \mathbb{R}$$

V v. space over $\mathbb{R}$.

Def A bilinear form on a real vector space V is skew symmetric if :

$$\left\{ \begin{array}{l} \varphi(u, v) = -\varphi(v, u) \\ \forall u, v \in V \times V \end{array} \right.$$

Rk (i) $\varphi(u, u) = -\varphi(u, u)$
 $\Rightarrow \varphi(u, u) = 0$

(ii) $\forall B$ basis of V ,

$$[\varphi]_B = -[\varphi]_B^T$$

(iii) $\forall A \in M_n(\mathbb{R})$

$$A = \underbrace{\frac{1}{2} (A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2} (A - A^T)}_{\text{skew symmetric}}$$

Thm

(Sylvester form) Let φ be a skew symmetric bilinear form over V (vector space over $\mathbb{R}$), then: $\exists$ a basis

$$\mathcal{B} = (v_1, w_1, v_2, w_2, \dots, v_m, w_m, v_{2m+1}, v_{2m+2}, \dots, v_n)$$

$\downarrow$
 $\dim V = n$
 $\mathbb{R} < +\infty$

s.t.:

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \\ & & & & & & & 0 \\ & & & & & & & & 0 \end{pmatrix}$$

Cor

Skew symmetric matrices have an even rank

proof (sketch) Induction on the dimension of V .

• $\varphi \equiv 0 \rightarrow \text{done}$

• $\varphi \neq 0 \Rightarrow \exists v_1, w_1 / \varphi(v_1, w_1) \neq 0$

• $v_1, w_1 \neq 0 \Rightarrow$ after scaling w_1 , we

can assume $\varphi(v_1, w_1) = 1$

$\Rightarrow \varphi(w_1, v_1) = -1$

• v_1, w_1 linearly independent

$\varphi(v_1, \lambda v_1) = \lambda \varphi(v_1, v_1) = 0$

• $U = \langle v_1, w_1 \rangle$

$W = \{ v \in V / \varphi(v_1, v) = \varphi(w_1, v) = 0 \}$

Exercise

$V = U \oplus W$

Now apply the induction hypothesis to $\varphi|_W$ $\square$

Inner product spaces

- definite positive bilinear forms
 - scalar product
 - norms : notion of distance
- $\Rightarrow$ spectacular infinite dimensional counterpart

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Def (Inner product) Let V be a vector space over $\mathbb{R}$ (resp. $\mathbb{C}$). An inner product on V is a positive definite symmetric (resp. Hermitian) form φ on V .

Notation $\langle u, v \rangle = \varphi(u, v)$

V is called a real (resp. complex) inner product space.

Examples

(i) $\mathbb{R}^n$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i$$

(ii) $\mathbb{C}^n$, $\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i}$

(iii) $V = \mathcal{C}([0,1], \mathbb{C})$,

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt$$

↑
def

" L^2 scalar
product"

(iv) We can fix a weight $w: [0,1] \rightarrow \mathbb{R}_+^*$

and define on $V = \mathcal{C}([0,1], \mathbb{C})$:

$$\langle f, g \rangle = \int_0^1 f(t) \bar{g}(t) \omega(t) dt.$$

$\Rightarrow$ One can check that these are inner products.

$$\left(\langle v, u \rangle = 0 \Rightarrow u = 0 \right) \text{ (definite)}$$

Def	(Norm / length)
$\ v\ = \left(\langle v, v \rangle \right)^{\frac{1}{2}}$	

norm derives
from a scalar
product

Rk $\langle v, v \rangle \in \mathbb{R}_+$, and

$$\|v\| = 0 \Rightarrow v = 0$$

↑
positive definite

1.1 : This allows us to define a notion of length.