


Lecture 2 Spans, linear independence and STEINITZ exchange lemma

→ dimension of a vector space
basis.

D.f (Span of a family of vectors)

let V be an F -vector space. let $S \subset V$
be a subset ($S \equiv$ set of vectors).

We define:

$\langle S \rangle = \left\{ \begin{array}{l} \text{finite linear combinations of} \\ \text{elements of } S \end{array} \right\}$

$\equiv$
Span of S

$$= \left\{ \sum_{s \in S} \overset{\in F}{\alpha_s} \overset{\in V}{s}, \quad s_s \in S, \text{ only finitely} \right\}$$

many $\rangle_s$ are non zero

Convention $\langle \emptyset \rangle = \{0\}$

Remark $\langle S \rangle \equiv$ smallest vector subspace of V which contains S .

Ex ① $V = \mathbb{R}^3$

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} \right\}$$

$$\Rightarrow \langle S \rangle = \left\{ \begin{pmatrix} a \\ b \\ b \end{pmatrix} \mid (a, b) \in \mathbb{R} \right\}$$

② $V = \mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_i \in \mathbb{R}, 1 \leq i \leq n \right\}$

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$$

$$V = \text{Span} (e_i)_{1 \leq i \leq n}$$

(3). X set, $V = \mathbb{R}^X = \{f: X \rightarrow \mathbb{R}\}$

$S_x: X \rightarrow \mathbb{R} \quad (x \in X)$

$$y \mapsto \begin{cases} 1 & \text{if } x=y \\ 0 & \text{otherwise} \end{cases}$$

$\text{Span} (S_x)_{x \in X}$

$$= \{f \in \mathbb{R}^X \mid f \text{ has finite support}\}$$

$$\left(\text{Supp } f = \{x \mid f(x) \neq 0\} \right)$$

Def

let V be an F -vector space.

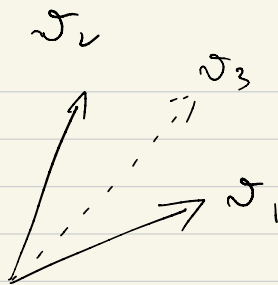
let S be a subset of V . We say

that S spans V if:

$$\langle S \rangle = V$$

Ex

$$V = \mathbb{R}^2$$



$$\langle \{v_1, v_2\} \rangle_{\mathbb{R}} = \mathbb{R}^2$$

$$\langle \{v_1, v_2, v_3\} \rangle_{\mathbb{R}} = \mathbb{R}^3$$

Def (Finite dimension)

Let V be an F -vector space. We say that V is finite dimensional if it is spanned by a finite set.

Example . $V = \mathbb{R}[x]$: polynomials on $\mathbb{R}$

$$V_n = \mathbb{P}_n[x] : \underline{\hspace{2cm}}$$

with degree $\leq n$, $n \in \mathbb{N}$.

$$V_n = \langle \{1, x, \dots, x^n\} \rangle$$

$\Rightarrow V_n$ is finite dimensional

Exercice $V = \mathbb{P}[x]$ is infinite dimensional
($\equiv$ not finite dimensional there is no family S with finitely many elements which span is V).

 If V is finite dimensional, is there a minimal number of vectors in the family required so that the family spans V ?

Def (Independence) We say that $(v_1, \dots, v_n)$ elements of V are LINEARLY INDEPENDENT if :

$$\sum_{i=1}^n \lambda_i v_i = 0 \quad \lambda_i \in F \quad \Rightarrow \quad \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$$

Equivalently, $(v_1, \dots, v_n)$ are not linearly independent if one of them is a linear combination of the $(n-1)$ remaining ones.

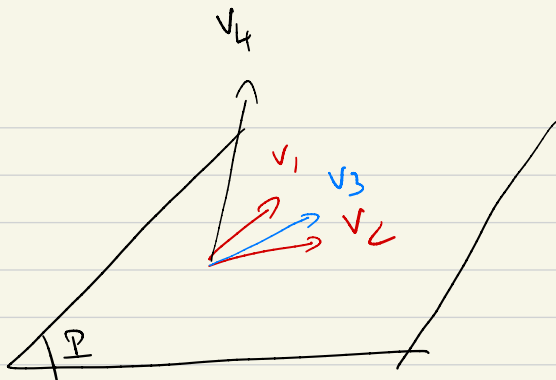
Indeed, $\exists (\lambda_1, \dots, \lambda_n), j \in \{1, n\} /$

$$\sum_{i=1}^n \lambda_i v_i = 0 \quad \lambda_j \neq 0$$

$$\Rightarrow v_j = -\frac{1}{\lambda_j} \sum_{\substack{i \neq j \\ 1 \leq i \leq n}} \lambda_i v_i$$

$\mathbb{F}_x$

$\mathbb{R}$



• (v_1, v_2) independent

• $v_3 \in \langle v_1, v_2 \rangle \Rightarrow (v_1, v_2, v_3)$ NOT linearly independent

• $v_4 \notin \langle v_1, v_2 \rangle \Rightarrow (v_1, v_2, v_4)$ are linearly independent.

Prop $(v_i)_{1 \leq i \leq n}$ linearly independent
 $\rightarrow \forall i \in \{1, n\}, v_i \neq 0$.

Def (Basis) A subset S of V is a
BASIS of V if :

(i) $\langle S \rangle = V$

(ii) S linearly independent

Rec When S spans V we say that S is a
generating family. S basis is a
linearly independent generating family.

Ex ① $V = \mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R}, 1 \leq i \leq n \right\}$

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th position}$$

$(e_i)_{1 \leq i \leq n}$ basis
for V .

Exhale.

② $V = \mathbb{C}$

. $\mathbb{C} = \mathbb{C}(=\mathbb{F})$ vector space, $\{1\}$ a basis

. $\mathbb{C} = \mathbb{R}(=\mathbb{F})$ ———, $\{1, i\}$ a basis

③ $V = \mathbb{R}[x] = \{ \mathbb{R}(x) \text{ polynomials on } \mathbb{R} \}$
 $S = \{x^n, n \geq 0\} \equiv \text{basis of } V.$

Lemma V $\mathbb{F}$ vector space. Then $(v_1, \dots, v_n)$ is a basis of V if and only if (iff) any vector $v \in V$ has a unique decomposition:

$$v = \sum_{i=1}^n \lambda_i v_i, \quad \lambda_i \in \mathbb{F}$$

Def $(\lambda_1, \dots, \lambda_n) \equiv \text{coordinates of } v \text{ in the basis } (v_1, \dots, v_n).$

proof. $\langle v_1, \dots, v_n \rangle = V$

$$\Rightarrow \forall v \in V, \exists (\lambda_1, \dots, \lambda_n) \in F^n /$$

$$v = \sum_{i=1}^n \lambda_i v_i$$

$$. \quad v = \sum_{i=1}^n \lambda_i v_i = \sum_{i=1}^n \lambda'_i v_i$$

$$\Rightarrow \sum_{i=1}^n (\lambda_i - \lambda'_i) v_i = 0$$

$$\Rightarrow \lambda_i = \lambda'_i, \quad \forall 1 \leq i \leq n$$

$(v_i)_{1 \leq i \leq n}$ linearly
independent

lemma If $(v_1, \dots, v_n)$ spans V ,
then some subset of this family is a
basis of V

proof If $(v_1, \dots, v_n)$ are linearly independent,
done. If they are not, then up
to changing indices,

$$v_n \in \text{Span}(v_1, \dots, v_{n-1})$$

(v_n is a linear combination of $v_1, \dots, v_{n-1}$)

$$\Rightarrow \langle v_1, \dots, v_n \rangle = \langle v_1, \dots, v_{n-1} \rangle$$

$\parallel$

$\vee$

$$\Rightarrow \langle v_1, \dots, v_{n-1} \rangle = V$$

I iterate this process

□

Th (STEINITZ EXCHANGE LEMMA)

Let V be a finite dimensional vector space over F . Take $(v_1, \dots, v_m)$ linearly independent, and $(w_1, \dots, w_n)$ which spans V . Then:

(i) $m \leq n$

(ii) Up to reordering,

$(v_1, \dots, v_m, w_{m+1}, \dots, w_n)$ spans V .

proof (Induction) Suppose that we have replaced l (≥ 0) of the w_i . Pending if necessary $\langle v_1, \dots, v_l, w_{l+1}, \dots, w_n \rangle = V$.

. $m = l$ done

. Assume $l < m$. Then : $v_{l+1} \in V$

$$g_{l+1} = \sum_{i \leq l} \alpha_i g_i + \sum_{i > l} \beta_i w_i$$

Since the $(v_i)_{1 \leq i \leq m}$ ($l+1 \leq m$)

linearly independent, then one of the β_i is non zero. Up to reordering:

$$w_{l+1} = \frac{1}{\beta_{l+1}} \left(v_{l+1} - \sum_{i \leq l} \alpha_i g_i - \sum_{i > l+1} \beta_i w_i \right)$$



$\Rightarrow V$ is spanned by

$g_1, \dots, g_{l+1}, w_{l+2}, \dots, w_n$

$\leadsto$ done after m steps

$\Rightarrow$ we must have replaced m w_i

$$\rightarrow \boxed{m \leq n}$$

