Lecture 19

Sylvester's law / Sesquilinear forms

Recall

Th $\dim_F V < +\infty$

$\varphi : V \times V \to F$ is a symmetric bilinear form,

then $\exists\, \mathcal{B}$ basis of V wrt φ is diagonal.

Cor $F = \mathbb{C}$, $\dim_{\mathbb{C}} V = n < +\infty$.

φ symmetric bilinear form of V

Then: $\exists\, \mathcal{B}$ basis of V st:

$$[\varphi]_{\mathcal{B}} = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right), \quad r = \operatorname{rk}(\varphi)$$

proof Pick a basis $\mathcal{E} = (e_1, \dots, e_n)$ such that:

$$[\varphi]_{\mathcal{E}} = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

Reorder a_i such that:

$$\begin{cases} a_i \neq 0 & \text{for } 1 \leq i \leq r \\ a_i = 0 & \text{for } i > r \end{cases}$$

For $i \leq r$, let $\sqrt{a_i}$ be a choice of (complex) root for a_i. Let:

$$v_i = \begin{cases} \dfrac{e_i}{\sqrt{a_i}} & 1 \leq i \leq r \\ e_i & i > r \end{cases}$$

$\mathcal{B} = (v_1, \dots, v_r, e_{r+1}, \dots, e_n)$ basis of V

$$[\varphi]_{\mathcal{B}} = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

□.

Cor Every symmetric matrix of $M_n(\mathbb{C})$ is congruent to a UNIQUE matrix of the form:

$$\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right).$$

Cor $F = \mathbb{R}$, $\left|\begin{array}{l} \dim_{\mathbb{R}} V = n < +\infty \\ \varphi \text{ symmetric bilinear form of } V \end{array}\right.$

Then: $\exists\, \mathcal{B} = (v_1, \dots, v_n)$ basis of V such that:

$$[\varphi]_{\mathcal{B}} = \left(\begin{array}{c|c|c} I_p & & 0 \\ \hline 0 & -I_q & \\ \hline & 0 & 0 \end{array}\right) \qquad \left|\begin{array}{l} p, q \geq 0 \\ p+q = r(\varphi) \end{array}\right.$$

proof $\mathcal{E} = (e_1, \dots, e_n)$,

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

Reorder indices so that:

$$\begin{vmatrix} a_i > 0, & 1 \le i \le p \\ a_i < 0, & p+1 \le i \le q \\ a_i = 0, & i > p+q \end{vmatrix}$$

We define:

$$V_i = \begin{cases} \dfrac{e_i}{\sqrt{a_i}} & 1 \le i \le p \\ \dfrac{e_i}{\sqrt{-a_i}} & p+1 \le i \le q \\ e_i & i > p+q \end{cases}$$

$\Rightarrow \mathcal{B} = (v_1, \dots, v_n)$ does the job $\square$

Def $s(\varphi) = p - q$

$\equiv$ signature of φ (or the associated Q)

This is well defined (independently of the basis).

Thm (Sylvester's law of inertia) $\left(\mathbb{F} = \mathbb{R},\ \dim_{\mathbb{F}} V = n < +\infty\right.$

If a real symmetric bilinear form is represented by:

$$\begin{pmatrix} I_p & & 0 \\ & -I_q & \\ 0 & & 0 \end{pmatrix} \quad \text{in } \mathcal{B} \text{ basis of } V$$

$$\begin{pmatrix} I_{p'} & & 0 \\ & -I_{q'} & \\ 0 & & 0 \end{pmatrix} \quad \text{in } \mathcal{B}' \text{ basis of } V$$

$$\Longrightarrow \quad p = p', \quad q = q'$$

Def φ symmetric bilinear form on a real vector space V. We say that:

- φ is positive definite $\Longleftrightarrow$ $\varphi(u,u) > 0$ $\forall u \in V \setminus \{0\}$
- φ is positive semi definite $\Longleftrightarrow$ $\varphi(u,u) \geq 0$ $\forall u \in V \setminus \{0\}$
- φ is negative definite $\Longleftrightarrow$ $\varphi(u,u) < 0$ $\forall u \in V \setminus \{0\}$
- φ is negative semi definite $\Longleftrightarrow$ $\varphi(u,u) \leq 0$ $\forall u \in V \setminus \{0\}$

Ex.

$$\begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} \Big\updownarrow n$$

- positive definite for $p = n$
- positive semi definite for $p \leq n$.

proof (Sylvester's law of inertia)

In order to prove uniqueness of p it is enough to show that p is the largest dimension of a subspace on which φ is definite positive.

- Say $\mathcal{B} = (v_1, \dots, v_n)$,

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & \boxed{-I_q} & 0 \\ 0 & 0 & 0 \end{pmatrix} \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\} \perp$$

Let $X = \langle v_1, \dots, v_p \rangle$. Then φ is positive definite on X: $u = \sum_{i=1}^{p} \lambda_i v_i$

$$Q(u) = \varphi(u, u) = \varphi\left(\sum_{i=1}^{p} \lambda_i v_i , \sum_{i=1}^{p} \lambda_i v_i \right)$$

$$= \sum_{i=1}^{p} \lambda_i^2$$

. Suppose that φ is definite positive on another subspace X'. Let:

$$X = \langle v_1, \dots, v_p \rangle, \quad Y = \langle v_{p+1}, \dots, v_n \rangle$$

Then we know (I look at the matrix in B) that φ is negative semi definite in Y

$$\Rightarrow \boxed{Y \cap X' = \{0\}}$$

$$\left(y \in Y \cap X' \underset{y \in Y}{\Rightarrow} Q(y) \leq 0 \underset{y \in X'}{\Rightarrow} y = 0 \right)$$

$$\Rightarrow \underbrace{\dim Y}_{(n-p)} + \dim X' \leq n$$

$$\Rightarrow \quad \dim X' \leq n - (n-p) = p$$

Similarly, q is the largest dimension of a subspace on which φ is definite negative $\square$.

Def $K = \{v \in V \,/\, \forall u \in V,\ \varphi(u,v) = 0\}$

$\shortparallel$

kernel of a bilinear form

Rk $\dim K + r(\varphi) = n$.

Rk Notation as above, $F = \mathbb{R}$.

There is a subspace T of dimension

$n-(p+q)+\min\{p,q\}$ such that :

$$\varphi_{|T}=0.$$

Indeed, say $p\geq q$, consider :

$$T=\langle \underbrace{v_1+v_{p+1},\dots,v_q+v_{p+q}}_{q},\underbrace{v_{p+q+1},\dots,v_n}_{n-(p+q)}\rangle$$

$$B=(v_1,\dots,v_n)$$

$$[\varphi]_B=\begin{pmatrix} I_p & & 0 \\ & -I_q & \\ 0 & & \boxed{0} \end{pmatrix}$$

Check $\varphi_{|T} = 0$

$$\left(\forall (u,v) \in T \times T, \ \varphi(u,v) = 0 \right)$$

Moreover, one can show that this is the largest possible dimension of a subspace T' such that $\varphi_{|T'} = 0$. □

Sesquilinear forms

$\boxed{F = \mathbb{C}}$

Standard inner product on $\mathbb{C}^n$ is:

$$x = \begin{vmatrix} x_1 \\ \vdots \\ x_n \end{vmatrix}, \quad x_i \in \mathbb{C}$$

$$y = \begin{vmatrix} y_1 \\ \vdots \\ y_n \end{vmatrix}, \quad y_i \in \mathbb{C}$$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

$$\left(\|x\|^2 = \langle x, x \rangle = \sum_{i=1}^{n} |x_i|^2 \right)$$

⚠

$$\mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$$

$$(x, y) \longmapsto \langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

is <u>NOT</u> a bilinear form: $\lambda \in \mathbb{C}$,

$$\left| \begin{array}{l} \langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle \\ \langle \lambda x, y \rangle = \lambda \langle x, y \rangle \end{array} \right.$$

$\boxed{\text{Def}}$ V, W are $\mathbb{C}$ vector spaces. A sesquilinear form is a function:

$\varphi : V \times W \longrightarrow \mathbb{C}$ s.t.:

(i) $\varphi(\lambda_1 v_1 + \lambda_2 v_2, w)$

$= \lambda_1 \varphi(v_1, w) + \lambda_2 \varphi(v_2, w)$

$(\forall \lambda_1, \lambda_2 \in \mathbb{C},\ \forall v_1, v_2 \in V,\ \forall w \in W)$

(linear with respect to the first coordinate)

(ii) $\varphi(v, \lambda_1 w_1 + \lambda_2 w_2)$

$= \overline{\lambda_1}\, \varphi(v, w_1) + \overline{\lambda_2}\, \varphi(v, w_2)$

$(\forall \lambda_1, \lambda_2 \in \mathbb{C},\ \forall v \in V,\ \forall w_1, w_2 \in W)$

(antilinear with respect to the second coordinate)

Def Notation as above.

$$\mathcal{B} = (v_1, \dots, v_m) \quad \text{of } V$$
$$\mathcal{C} = (w_1, \dots, w_n) \quad \text{of } W$$
$$[\varphi]_{\mathcal{B},\mathcal{C}} = \big(\varphi(v_i, w_j)\big)_{m \times n}$$

Lemma $\varphi(v, w) = [u]_{\mathcal{B}}^T \, [\varphi]_{\mathcal{B},\mathcal{C}} \, \overline{[v]_{\mathcal{C}}}$

proof Exercise, similar to the bilinear case.

Lemma $\mathcal{B}, \mathcal{B}'$ basis for V, $P = [\mathrm{Id}]_{\mathcal{B}',\mathcal{B}}$

$\mathcal{C}, \mathcal{C}'$ ——— W, $Q = [\mathrm{Id}]_{\mathcal{C}',\mathcal{C}}$

Then $[\varphi]_{\mathcal{B}',\mathcal{B}} = P^T \, [\varphi]_{\mathcal{B},\mathcal{C}} \, \overline{Q}$

proof Analogous to the bilinear case □.