Lecture 18 — Bilinear forms

Bilinear form $\varphi : V \times V \to F$ bilinear form.

- $\dim_F V < +\infty$, $\mathcal{B}$ basis of V
- $[\varphi]_{\mathcal{B}} = [\varphi]_{\mathcal{B},\mathcal{B}} = \left(\varphi(e_i, e_j)\right)_{1 \le i,j \le n}$
  $\mathcal{B} = (e_i, e_j)$

$$\boxed{\begin{array}{l} \textbf{Lemma} \quad \varphi : V \times V \to F \text{ bilinear} \\ \quad \mathcal{B}, \mathcal{B}' \text{ basis for } V, \quad P = [\mathrm{Id}]_{\mathcal{B}', \mathcal{B}} \\ \Longrightarrow [\varphi]_{\mathcal{B}'} = P^T [\varphi]_{\mathcal{B}} P \end{array}}$$

proof Special case of general formula of Lecture 10. $\square$

Def $A, B \in M_n(F)$ are congruent if $\exists P$ invertible such that: $A = {}^tPBP$.

Rk This is an equivalence relation.

Def A bilinear form φ on V is symmetric if:

$$\varphi(u, v) = \varphi(v, u) \quad \forall u, v \in V$$

Remarks

- $A \in M_n(F)$, we say that

$$A \text{ symmetric} \iff A = A^T$$

$$\iff \left[A = (a_{ij})_{1 \leq i,j \leq n},\ a_{ij} = a_{ji}\right]$$

- φ is symmetric $\iff [\varphi]_B$ is symmetric in <u>ANY</u> basis B.

. To be able to represent φ by a diagonal matrix, you must be symmetric:

$$P^T A P = D \Rightarrow D^T = P^T A^T P$$

D ↑ diagonal, $D^T = D$

$$\Rightarrow A = A^T$$

Def A map $Q : V \to F$ is a quadratic form iff there exist a bilinear form $\varphi : V \times V \to F$ such that:

$$Q(u) = \varphi(u, u)$$

Remark $\mathcal{B} = (e_i)_{1 \le i \le n}$

$$A = [\varphi]_{\mathcal{B}} = \big(\underbrace{\varphi(e_i, e_j)}_{a_{ij}}\big)_{1 \le i, j \le n}$$

Then: $u = \sum_{i=1}^{n} x_i e_i$ $= X$, then:

$$Q(u) = \varphi\left(\sum_{i=1}^{n} x_i e_i , \sum_{i=1}^{n} x_i e_i \right)$$

$$= \sum_{i,j=1}^{n} x_i x_j \varphi(e_i, e_j)$$

$\uparrow$ φ bilinear

$$= \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

$$= X^T A X, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A = (a_{ij})$$

$\uparrow$ direct check

$$\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \cdot$$

$$\boxed{Q(X) = X^T A X.}$$

. Observe $\quad X^T A X = \sum_{i,j=1}^{n} a_{ij} x_i x_j$

$$= \sum_{i,j=1}^{n} a_{ji} x_i x_j$$

$$= \frac{1}{2} \sum_{i,j=1}^{n} (a_{ij} + a_{ji}) x_i x_j$$

$$= \frac{1}{2} X^T (\underbrace{A + A^T}_{\text{symmetric}}) X$$

$$\boxed{\begin{array}{l}\boxed{\text{Prop}} \quad \text{If } Q : V \times V \to F \text{ is a quadratic form,} \\ \text{then there exists a unique symmetric} \\ \text{bilinear form } \varphi : V \times V \to F \text{ such that:} \\ \qquad Q(u) = \varphi(u, u) \quad \forall u \in V\end{array}}$$

proof . Let φ bilinear form on V /

$\forall u,\ Q(u) = \psi(u,u)$. Let:

$$\varphi(u,v) = \frac{1}{2}\underbrace{\left(\psi(u,v) + \psi(v,u)\right)}, \quad \text{then:}$$

- φ symmetric
- $\varphi(u,u) = \psi(u,u) = Q(u)$

($\rightsquigarrow$ in a basis, $A \longrightarrow A + A^T$)

- <u>Uniqueness</u> φ is a symmetric bilinear form, then: $\forall (u,v) \in V$,

$$
\begin{aligned}
Q(u+v) &= \varphi(u+v, u+v) \\
&= \varphi(u,u) + \varphi(u,v) + \varphi(v,u) + \varphi(v,v) \\
&= Q(u) + 2\varphi(u,v) + Q(v)
\end{aligned}
$$

$$\Rightarrow \boxed{\varphi(u,v) = \frac{1}{2}\left[Q(u+v) - Q(u) - Q(v)\right]}$$

$\equiv$ POLARIZATION IDENTITY.

D.

Thm (Diagonalization of symmetric bilinear forms)

Let $\varphi : V \times V \longrightarrow F$ be a symmetric bilinear form, $(\dim_F V < +\infty)$. Then there exist a basis of V such that:

$$[\varphi]_{\mathcal{B}} \text{ is diagonal.}$$

proof Induction on dimension.

- $n = 0, 1$ ✓.
- Suppose Thm holds for all dimension $< n$.
- If $\varphi(u, u) = 0 \ \forall u \in V \Longrightarrow \varphi$ is identically zero

polarization identity

$\Longrightarrow$ done.

. $\varphi \neq 0$, I can always find $u \in V \setminus \{0\}$ / $\varphi(u, u) \neq 0$.

Let us call $u = e_1$. Let us define:

$$U = (\langle e_1 \rangle)^{\perp} \underset{\text{def}}{=} \{ v \in V \,/\, \varphi(e_1, v) = 0 \}$$

$\shortparallel$

orthogonal

$$= \operatorname{Ker} \left\{ \varphi(e_1, \cdot) : \begin{array}{l} V \to F \\ v \mapsto \varphi(e_1, v) \end{array} \right\}$$

Rank nullity $\dim V = n = \underbrace{1}_{\operatorname{rk}(\varphi(e_1, \cdot))} + \dim U$

$\varphi(e_1, e_1) \neq 0$

$$\Rightarrow \boxed{\dim U = n - 1}$$

. $U + \langle e_1 \rangle = U \oplus \langle e_1 \rangle$

Indeed, $v \in \langle e_1 \rangle \cap U$

$$\Rightarrow \left| \begin{array}{l} v = \lambda e_1 \\ \lambda \in F \end{array} \right. , \quad \varphi(e_1, v) = 0$$

$$\varphi(e_1, v) = \varphi(e_1, \lambda e_1) = \lambda \underbrace{\varphi(e_1, e_1)}_{\neq 0}$$

$$\Rightarrow \lambda = 0 \Rightarrow v = 0.$$

$$U + \langle e_1 \rangle = U \oplus \langle e_1 \rangle$$

$\dim U = n-1$ $\qquad$ $\dim 1$

$$\Rightarrow \boxed{V = \langle e_1 \rangle \oplus U} \Leftarrow \text{FUNDAMENTAL}$$

. <u>Pick</u> $\left| \begin{array}{l} (e_2, \dots, e_n) \text{ basis of } U \\ \mathcal{B} = (e_1, \dots, e_n) \text{ basis of } V \end{array} \right.$

and :

$$[\varphi]_B = \begin{array}{c} \left(\begin{array}{c|c} \varphi(e_1, e_1) & * \\ \hline 0 & \\ \vdots & A' \\ 0 & \end{array} \right) \end{array} \begin{array}{c} e_1 \\ \vdots \\ \vdots \\ e_n \end{array}$$

$$A' = \left(\varphi(e_i, e_j)\right)_{\substack{2 \le i \le n \\ 2 \le j \le n}} \qquad A' = {}^T A'$$

$$A' = [\varphi]_{B'} \qquad B' = (e_2, \dots, e_n)$$

$\varphi|_U : U \times U \to F$ is symmetric with matrix A'

$\Rightarrow$ apply the induction hypothesis on U

$\Rightarrow$ find $(e_2', \dots, e_n')$ such that

$\varphi|_{(e_1', \dots, e_n')}$ diagonal

$\Rightarrow$ φ diagonal in $(e_1, e_2', \dots, e_n')$ $\square$

Example $V = \mathbb{R}^3$, e_1, e_2, e_3.

- $Q(x_1, x_2, x_3) = 1x_1^2 + 1x_2^2 + 2x_3^2 + 2x_1x_2$
  $+ 2x_1x_3 - 2x_2x_3$

  $= X^T A X$, $\boxed{A = A^T}$.

- $$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

Diagonalize it

(1) Follow the proof of diagonalization
$\rightsquigarrow$ algorithm

② "Complete the square":

$$Q(x_1, x_2, x_3) = \cancel{x_1^2} + \cancel{x_2^2} + 2x_3^2 + \cancel{2x_1 x_2} + \cancel{2x_1 x_3} - 2x_2 x_3$$

$$= (x_1 + x_2 + x_3)^2 + x_3^2 - 4x_2 x_3$$

$$= \underbrace{(x_1 + x_2 + x_3)}_{x_1'}{}^2 + \underbrace{(x_3 - 2x_2)}_{x_2'}{}^2 - \underbrace{(2x_2)}_{x_3'}{}^2$$

$$\Rightarrow P, \quad P^T A P = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -1 \end{pmatrix} \to -4$$

To find P, notice that:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underbrace{\boxed{\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & 0 \end{pmatrix}}}_{P^{-1}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

D.