Lecture 17 Jordan normal form, examples

Today, $F = \mathbb{C}$.

Def (Jordan normal form)

Let $A \in M_n(\mathbb{C})$, we say that A is in Jordan Normal Form (JNF) if it is a block diagonal matrix:

$$A = \begin{pmatrix} J_{n_1}(\lambda_1) & & & 0 \\ & J_{n_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{n_k}(\lambda_k) \end{pmatrix}$$

where: . $k \geq 1$, $n_1, \dots, n_k$ integers,

$$\sum_{i=1}^{k} n_i = n.$$

- $\lambda_i \in \mathbb{C}$, $1 \le i \le k$ (they need not be distinct)
- $m \geq 1$, $\lambda \in \mathbb{C}$.

$$J_m(\lambda) = \begin{cases} (\lambda) & \text{for } m = 1 \\ \begin{array}{c} \\ \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \\ \longleftarrow m \longrightarrow \end{array} \!\!\! \begin{array}{c} \uparrow \\ m \\ \downarrow \end{array} & \text{for } m \geq 2 \end{cases}$$

So $J_m \in M_m(\mathbb{C}) \equiv$ Jordan block.

Remark $n=3$, $A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$

$$= \begin{pmatrix} J_1(\lambda) & & 0 \\ & J_1(\lambda) & \\ 0 & & J_1(\lambda) \end{pmatrix} \equiv \text{JNF}.$$

Thm Every matrix $A \in M_n(\mathbb{C})$ is similar to a matrix in JNF, which is unique up to reordering the Jordan blocks.

proof Non examinable

($\Rightarrow$ follows from the main Theorem in the Group - Ring - Modules class) $\square$

Ex **$n=2$** Possible JNF in this case?

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$m_A = (t-\lambda_1)(t-\lambda_2)$ $\qquad m_A = (t-\lambda)$ $\qquad m_A = (t-\lambda)^2$

$\lambda_1 \neq \lambda_2$

$n=3$

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} \qquad (t-\lambda_1)(t-\lambda_2)(t-\lambda_3)$$

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_2 \end{pmatrix} \qquad (t-\lambda_1)(t-\lambda_2)$$

$$\begin{pmatrix} \boxed{\lambda_1} & 0 & 0 \\ & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix} \qquad (t-\lambda_1)(t-\lambda_2)^2$$

$$\begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda \end{pmatrix} \qquad (t-\lambda)$$

$$\left(\begin{array}{c|cc} \lambda & 0 & \\ \hline 0 & \lambda & 1 \\ & 0 & \lambda \end{array}\right) \qquad (t-\lambda)^2$$

$$\begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ 0 & & \lambda \end{pmatrix} \qquad (t-\lambda)^3$$

Thm (Generalized eigenspace decomposition)

V vector space over $\mathbb{C}$, $\dim_{\mathbb{C}} V < +\infty$

$\alpha \in L(V)$ with:

$$m_\alpha(t) = (t-\lambda_1)^{c_1} \dots (t-\lambda_\ell)^{c_\ell}$$

$(\lambda_i)_{1 \le i \le \ell}$ distinct eigenvalues

Then $$V = \bigoplus_{i=1}^{\ell} V_j \qquad (*)$$

$$V_j = \operatorname{Ker}\left[(\alpha - \lambda_j \mathrm{Id})^{c_j}\right]$$

$$V_j = \mathrm{Ker}\left[\left(\alpha - \lambda_j \mathrm{Id}\right)^{c_j}\right] \equiv \text{generalized eigenspace.}$$

Rk When α diagonalizable, $c_j = 1$,

$$\left| \begin{array}{l} V_j = \mathrm{Ker}\left(\alpha - \lambda_j \mathrm{Id}\right) \\ V = \bigoplus_{j=1}^{k} V_j \end{array} \right.$$

proof Projectors onto V_j are explicit. Indeed:

$$p_j(t) = \prod_{i \neq j} (t - \lambda_i)^{c_i}.$$

Then the p_j have NO common factor so by Euclid's algorithm, we can find $q_1, \ldots, q_k$ polynomials over $\mathbb{C}$ s.t.:

$$\sum_{i=1}^{k} p_i q_i = 1.$$

Let us define : $\pi_j = q_j p_j(\alpha)$.

(i) $\displaystyle\sum_{j=1}^{k} \pi_j = \mathrm{Id}$ by construction

$\forall v \in V, \quad v = \displaystyle\sum_{j=1}^{k} \underbrace{\pi_j(v)}_{\in V_j}$.

(ii) We know $m_\alpha(\alpha) = 0$, so :

$$(\alpha - \lambda_j \mathrm{Id})^{c_j} \pi_j = 0$$

$$\Rightarrow \quad \mathrm{Im}\, \pi_j \subset V_j$$

$$\Rightarrow \quad V = \sum_{j=1}^{k} V_j$$

(iii) Sum is direct.

$$\left| \begin{array}{l} \pi_i \pi_j = 0 \quad \text{if } i \neq j \\ \pi_i = \pi_i \underbrace{\left(\displaystyle\sum_{j=1}^{k} \pi_j \right)}_{\mathrm{Id}} = \pi_i^2 \end{array} \right.$$

(projector property)

So: $\pi_i|_{V_j} = \begin{cases} Id & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

from which the direct sum property follows. $\square$

<u>Rk</u> This decomposition can be used to reduce the proof of JNF to a single eigenvalue.

($\rightsquigarrow$ study of "nilpotent matrices",

A nilpotent $\Longleftrightarrow \exists n \in \mathbb{N} \,/\, A^n = 0$)

<u>Rk</u> We can compute on the JNF the quantities a_λ, g_λ, c_λ. Indeed,

let $m \geq 2$ and consider:

$$J_m(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

$$J_m - \lambda Id = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

$$(J_m - \lambda Id)^2 = \begin{pmatrix} 0 & 0 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & 0 & & \ddots & 0 \\ & & & & 0 \end{pmatrix}$$

By induction, one can show:

$$(J_m - \lambda \mathrm{Id})^k = \begin{pmatrix} 0 & I_{m-k} \\ 0 & 0 \end{pmatrix} \quad k < m$$

$$= 0 \quad \text{for } k \geq m$$

We say that $(J_m - \lambda \mathrm{Id})$ is nilpotent of order m

- $a_\lambda \equiv$ sum of sizes of blocks with eigenvalue λ

characteristic polynomial of a JNF $\equiv \# \lambda$ on the diagonal

- $g_\lambda \equiv \#$ blocks with eigenvalue λ.

$\underline{\cdot \; \mathbb{C}_{>}}$ $\quad (\lambda) \qquad t - \lambda$ kills it

$\qquad J_m(\lambda) \qquad (t-\lambda)^m$

$\equiv$ size of the largest block with eigenvalue λ.

Example

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$$

Find a basis in which A is in JNF.

$\underline{(i)}$ $$\chi_A(t) = \begin{vmatrix} -t & -1 \\ 1 & 2-t \end{vmatrix} = -t(2-t) + 1$$
$$= t^2 - 2t + 1$$
$$= (t-1)^2$$

$\Rightarrow$ one eigenvalue $\lambda = 1$.

$A - \mathrm{Id} \neq 0 \implies m_A(t) = (t-1)^2$

$\implies$ $\underline{\text{JNF}}$ $\quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(ii) Eigenvectors :

. $A - Id = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$

$\ker(A - Id) = \langle v_1 \rangle, \quad v_1 = \begin{vmatrix} 1 \\ -1 \end{vmatrix}$

. I look for v_2 (non unique !) such that :

$$\boxed{(A - Id) v_2 = v_1}$$

$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \Big| \; v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ works

$\Big| \; \mathrm{Mat}_{[B]} A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = Id + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$B = (v_1, v_2)$

$$\underline{P}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}}_{\underline{P}^{-1}} \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{J} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}^{-1}}_{\underline{P}}$$

$\Rightarrow$ this is how we find such a basis.

Exercise $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$\Longrightarrow$ find a basis in which A is JNF.